Differential Galois Theory for some Spectral Problems

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Madrid, 2018

A Mónica,
mi ancla con la realidad

Daría todo lo que sé por la mitad de lo que ignoro
R. Descartes

Agradecimientos

Han pasado ya cuatro años desde que me embarqué en esta aventura y llegado el final puedo asegurar que, aunque no ha sido fácil, he disfrutado cada día de ella.

En primer lugar, me gustaría agradecer a mis directores de tesis, Juan José Morales y María Ángeles Zurro, todo el tiempo que han pasado conmigo y todo lo que me han enseñado estos años. Gracias por ser una fuente inagotable de ideas, por estar disponibles cada vez que tenía una duda, hacer que me sintiera cómoda hablando de matemáticas con vosotros y sacar lo mejor de mí. En definitiva, gracias por convertirme en matemática.
Marian, a ti tengo que agradecerte especialmente por hacerme (¡hace ya seis años!) el regalo de la Teoría de Galois Diferencial y por dejarme emprender contigo un camino cuya tercera etapa concluye aquí.

Gracias a Sonia Jiménez por tener siempre un hueco para hacer cálculos en Mathematica. Por venir desde Salamanca los viernes y quedarse muchos de ellos hasta las tantas.

Muchas gracias al grupo DGTUAI: Rafael Hernández, Sonia Jiménez, José Rojo, Sonia L. Rueda y más recientemente José Ángel Capitán y Álvaro Pérez. Gracias por hacerme sentir una más desde el primer día y por enseñarme que las matemáticas se hacen en familia.

Durante el doctorado tuve la oportunidad de disfrutar de una estancia de investigación de tres meses financiada por la Universidad Autónoma de Madrid. Gracias a esta estancia pude conocer Limoges, una ciudad preciosa y acogedora a la que estoy deseando volver. Y sobre todo, pude conocer a gente maravillosa que me aportó mucho, tanto en el ámbito profesional como en el personal.

Merci beaucoup à Moulay Barkatou et Jacques-Arthur Weil, et, en général, à tout le groupe de Calcul Formel, pour m'avoir accueilli dans XLIM les trois mois que j'étais là. Jacques-Arthur et Moulay, merci de m'avoir appris une autre façon de travailler. J'ai beaucoup appris en travaillant avec vous et j'ai été très à l'aise.
Et merci de tout mon coeur à Ali, Joelle, Simone et Vincent pour me faire sentir à la maison ces trois mois que j'ai passé à Limoges. Merci pour le tenis, les bières et les gateaux.

Primitivo Acosta, Primi, tuve el placer de conocerte no hace mucho. Tu trabajo, junto con Jacques-Arthur y Moulay, ha sido el motivo del último capítulo de esta tesis. Gracias por dejarme formar parte de él.

During these years I have had the opportunity to meet many people in the conferences that I have attended. I would like to highlight Emma Previato among them, who shared her work with me and was interested in mine. Thank you for facilitating, in addition, my participation in the Seventh Iberoamerican Congress on Geometry in Valladolid.

Llegado este momento, no puedo evitar acordarme de dos personas que fueron determinantes en mi formación académica preuniversitaria para que decidiera, hace mucho ya, estudiar matemáticas: Lola y Juanito. Este trabajo es en parte vuestro.

No quisiera terminar esta parte sin agradecerle a Ana Bravo, mi tutora del grado, que siempre tuviera su puerta abierta para mi y me facilitara la vida todo lo que estuvo en su mano durante mis años de carrera.

Finally, I would like to thank Moulay Barkatou and David Blazquez for accepting to write the international reports of this thesis.

En el ámbito personal, me gustaría dar las gracias a mis compañeros de departamento. Muchas gracias a los que se fueron pronto, pero que siempre vuelven: María, Jose, Adri y Javi. Gracias por ser un ejemplo y una motivación para mí cuando empecé la tesis. Gracias a los que han estado la mayor parte del tiempo: Bea P., Irina, Marcos, Carlos, David, Iason, Leyter, Marta, Nikita, Jaime, Martí, Sonja, Manuel Jesús y José Luis (espero no dejarme a nadie). Gracias por hacer cada comida y cada descanso agradable y divertido. Por ser mi familia estos años y compartir alegrías y penas. Y por último, gracias a los que llegaron hace poco, pero que parece que siempre estuvieron aquí: Diego, Álex, Fran y Adri.

Especial mención merece el 103. Muchas gracias a Felipe, Dani, Ale, Bea y Julio por compartir conmigo cada día de esta aventura. Julio, tú has sido el último en llegar, pero pronto te has convertido en un amigo muy especial. Espero que lo que te queda de tesis te vaya muy bien. Felipe, gracias por incluirme en todos tus proyectos. Eres una de las personas con las que mejor me lo paso hablando. Dani, gracias por escucharme cuando más lo necesitaba y estar dispuesto a todo por arrancarme una sonrisa. Ale, gracias por todas las comidas y cenas a las que nos has invitado en tu casa. Y gracias por venir al despacho todos los días con una sonrisa. Por último, Bea, mi hermana de burocracia de tesis. Empezamos la tesis juntas y la vamos a acabar juntas. Hemos recorrido juntas todos y cada uno de los pasos de este camino y hemos compartido cada subida y bajada de esta montaña rusa emocional que ha resultado ser la tesis. Te has convertido en mucho más que una compañera de despacho. Gracias.

Por último, quiero dar las gracias a mis padres porque sin ellos jamás hubiera llegado hasta aquí. Gracias por enseñarme a confiar en mi misma, por darme todo lo que está a vuestro alcance y más, por enseñarme el valor del conocimiento, por respetar mis decisiones y, sobre todo, por estar siempre preparados para recogerme cuando me he caído. Y a mi hermana, Mónica. Tú sabes perfectamente todo lo que significas para mí, así que aquí sólo te diré gracias.

Contents

Introduction: summary and conclusions

Historical framework

Differential Galois Theory

In 1883, E. Picard published the article [73] in which he laid the foundations of the Differential Galois Theory, a Galois Theory for linear differential equations. This work was followed shortly after by the articles [74, 75, 87], where E. Picard and his Ph.D. student E. Vessiot consolidated this new theory. For this reason it is also known as Picard–Vessiot Theory.

Some years later, E. Kolchin wrote the works of E. Picard and E. Vessiot in modern language and obtained new important results, see [56, 57, 58]. In particular, in [58] he showed how to extend the Picard–Vessiot Theory to differential fields with partial derivatives. I. Kaplanski, in his magnificent book [54], facilitated the dissemination of Kolchin's work.

More recenlty, in 1986, J. J. Kovacic presented an algorithm, the so-called Kovacic's algorithm, for solving second order linear differential equations in closed form based on a theorem of classification of subgroups of $SL(2, \mathbf{C})$, see [59].

J.-P. Ramis was the first to find a connection between the irregular singularities of linear differential equations and Picard–Vessiot Theory. In particular, he proved that the Stokes matrices associated to this kind of singularities belong to the differential Galois group (see [20, 66, 77, 78, 79]).

N. Katz and P. Deligne introduced the Tannakian approach to Differential Galois Theory, see [31, 55]. In [64], M. Loday established the link between the Tannakian and analytical approaches given by the previous authors to these problems of Differential Galois Theory.

In recent years new differential Galois theories have been developed. We would like to highlight the works of B. Malgrange and H. Umemura, see [65, 86], on non-linear

Differential Galois Theory; P. J. Cassidy and M. F. Singer, see [21], on Parameterized Picard–Vessiot Theory; and the works of J. J. Morales-Ruiz and J.-P. Ramis on Differential Galois Theory for hamiltonian systems, also known as Morales–Ramis Theory, see for instance [69, 70].

In this Thesis we focus on the classical Picard–Vessiot Theory. Some recent references for this approach are [20, 23, 24, 84].

Darboux transformations

In 1882, G. Darboux presented in [26] a kind of transformations for second order linear differential equations which allowed to obtain solutions of these differential equations (see also [27, 28]). Some years later, in 1955, M. Crum described an iterative method to apply Darboux transformations, see [25]. For this reason, in this Ph.D. thesis we call these transformations *Darboux–Crum transformations*.

In the last years, Darboux transformations have been studied from different points of view by many authors, we refer to [67] and references therein. We also highlight here the work of P. Deift, who made a quite general study of the Darboux transformations from an spectral point of view, see [30].

There are two other approaches equivalent to Darboux transformations. The first of them is the factorization of operators. In 1941, E. Schrödinger showed that there are many ways of factoring the hypergeometric equation, see [81]. Ten years later, L. Infeld and T. E. Hull presented in [46] their factorization method, where they classified the factorizations of second order linear differential equations for eigenvalue problems in wave mechanics.

The other approach is in the context of supersymmetric quantum mechanics. According to the book of V. B. Matveev and M. A. Salle ([67]), the relation between supersymmetric quantum mechanics and the Darboux transformations was discovered by A. Andrianov, N. Borisov and M. Ioffe, see [9]. For this approach, we refer to the seminal works of E. Witten and L. Gendenshteïn, see [89] and [39] respectively. And to the Ph.D. thesis [4] and references therein, in which a galoisian approach to Darboux transformations and shape invariant potentials was also presented.

In this work we study the galoisian invariance of the Darboux–Crum transformations.

Solitonic partial differential equations

In 1967, C. Gardner, J. Greene, M. Kruskal and R. Miura found a method for solving some relevant initial value problem for the Korteweg–de Vries equation, see [36]. There, they related these solutions with the inverse scattering problem for the Schrödinger equation, proving that the eigenvalues of the Schrödinger equation are integrals of the Korteweg–de Vries equation. This was the begining of the Theory of Solitons.

One year later, C. Gardner, M. Kruskal and R. Miura published another paper ([37]) where they derived several conservation laws and constants of motion of the Korteweg–de Vries equation. The same year and based on their works, P. Lax published [60], where he used for the first time the commutator of two differential operators to describe certain evolution equations, nowadays known as Lax pairs. Then, P. Lax made other relevant contributions to Korteweg–de Vries equation, see [61, 62].

In their seminal work [1], M. J. Ablowitz, D. J. Kaup, A. C. Newell and H. Segur associated a linear differential system to some important families of evolution equations of solitonic type, nowadays called AKNS system after them.

H. Airault, H. McKean and J. Moser found conditions for rational and elliptic functions to be solutions of the Korteweg–de Vries equation, see [8]. Some years later, M. Adler and J. Moser constructed in [7] a family of rational solutions of the Korteweg–de Vries equations.

J. Duistermaat and F. Grünbaum in [32] made a differential study of the spectral parameter of the Schrödinger equation and attempted to obtain conditions to determine all the time independent rational solutions of the Korteweg–de Vries equations.

More recently, F. Gesztesy, K. Unterkofler and R. Weikard characterized the three classes of algebro-geometric solutions of the KdV hierarchy (rational, meromorphic simply periodic and elliptic) by means of Calogero–Moser systems, see [41].

Our approach to the study of solitonic equations is fitted within the aforementioned approaches.

State of the art

Throughout these last years many works have been published in which some of the previously discussed topics were related. Yu.V. Brezhnev applied the Picard–Vessiot Theory to the algebro–geometric (or finite gap) solutions of the solitonic equations, see [16, 17, 18]. A galoisian approach to Darboux transformations in one dimensional supersymmetric quantum mechanics was given in [4], see also [2, 3].

E. Horozov, using the work of J. Duistermaat and F. Grünbaum, studied in [44] the bispectral problem and described all bispectral operators contained in the Weyl algebra and their Darboux transformations, the so-called bispectral Darboux transformations.

Recently, the Differential Galois Theory have been applied in a very original way to the study of integrability of the polynomial vector fields in the plane, see [5, 6], and to the problem of finding closed form solutions of the Schrödinger equation for stationary KdV potentials, see [71]. In this last work, the authors also studied for the first time the differential extensions given by these solutions by means of the spectral parameter, the Spectral Picard–Vessiot Theory. In this thesis we will not deal with spectral extensions, but they will be key to understand the behaviour of the differential extensions that we may obtain without fixing the spectral parameter.

In the one dimensional case, the school of Kyoto has studied linear differential equations in the spectral parameter and a general theory of monodromy preserving deformations is developed for a system of linear ordinary differential equations. See the works of M. Jimbo, T. Miwa and K. Ueno [48, 49, 50].

This year T. Combot in [22] has given a complete classification of rational integrable potentials for the one-dimensional Schrödinger equation with spectral parameter.

Our work on the Schrödinger equation is fitted for the $1+1$ dimensional case, with special attention to rational potentials.

Structure and Contents

This Ph.D. thesis is structured in six chapters. The guiding ideas of all of them are the galoisian invariance under Darboux transformations and the time invariance of the differential Galois groups of the differential systems studied. Next I will briefly discuss the contents of each one, highlighting in each case the main original results.

Chapter 1

Along the first chapter, we shall give the preliminary concepts and results concerning integrable systems, differential operators, differential Galois Theory and the Schrödinger equation needed for the development of our work, which will be exposed in the following five chapters. In this chapter there are no original results.

Chapter 2

The second chapter is devoted to study how the differential Galois groups of the linear systems of (Zhakharov–Shabat) AKNS type behave under matrix Darboux transformations. An AKNS system is an integrable system of partial differential equations, introduced by and named after M. J. Ablowitz, D. J. Kaup, A. C. Newell and H. Segur (AKNS) in 1974 (see [1]). The zero curvature conditions of such systems become non linear partial differential equations. For instance they include all the classical solitonic hierarchies as the KdV hierarchy, an infinite sequence of partial differential equations which starts with the Korteweg–de Vries equation.

A Darboux transformation for an AKNS system is a gauge transformation which is covariant with respect to the AKNS structure. We are specially interested in the differential extension that Darboux transformations produce in the field of coefficients of the AKNS system.

The main result of this part is Theorem 2.2, which states that the differential Galois group of the transformed system by means of a Darboux transformation is isomorphic to a subgroup of the differential Galois group of the initial system. Thus, from our point of view this means that the transformed system is at least as integrable (in the Picard–Vessiot sense) as the initial system was. In particular, if the initial system is integrable, the transformed system is integrable too, in complete agreement with Darboux ideas (see [28, p. 10]).

We will obtain the isomorphism between the identity connected component of both differential Galois groups when the field of coefficients of the transformed system is an algebraic extension of the field of coefficients of the initial system (see Corollary 2.5).

As far as we know, this is the first time that the AKNS systems are approached from a galoisian point of view. All the results stated along this chapter, unless otherwise indicated, are considered original results.

The results contained in this chapter were published in our article [51] and have been presented in the following conferences and seminars:

- International Conference on Algebraic Methods in Dynamical Systems (AMDS 2018), Technical University of Madrid, June 2018.
- Functional Equations in Limoges (FELIM 2018), Research Center XLIM, Limoges (France), March 2018.
- Computational Algebra Seminar, XLIM, Limoges (France), March 2018.
- Finite Dimensional Integrable Systems in Geometry and Mathematical Physics (FDIS 2017), Centre de Recerca Matemàtica (CRM), Barcelona, July 2017.
- Geometry of Singularities and Differential Equations (GSDS 2017), University of Cantabria, June 2017.
- Seminar of Integrability and Differential Galois Theory, Technical University of Madrid, April 2016.

Chapter 3

Throughout Chapter 3 we develop the KdV hierarchy for both the stationary case and the time dependent case ($1+1$ dimensional case). These non linear differential equations are obtained as zero curvature conditions of a family of integrable systems by means of the spatial derivative, say ∂_x, of certain differential polynomials f_j defined recursively (see (3.1)).

For stationary potentials, we consider the spectral curve Γ associated to the Schrödinger operator

$$\mathcal{L} - E = -\partial_{xx} + u - E$$

at each level of the stationary KdV hierarchy. Usually this curve is considered as a plane algebraic curve and it can be a singular plane curve. In our approach to the Darboux transfomations for this type of systems we have needed to consider Γ but in $\mathbb{P}^2$, the complex projective plane. Hence we included in our analysis the point of infinity of these curves.

We study how Γ changes when we perform a classical Darboux–Crum transformation to the Schrödinger operator. The action of the tranformation strongly depends on the value of E we chose to compute it. Theorem 3.16 establishes this behaviour. We find that, when the value of E corresponds to a regular point of the curve, the spectral curve remains invariant. Otherwise, the new curve is a blowing-up or blowing-down of the initial curve depending on the type of Darboux transformation we perform.

The diagonal Green's function will be an essential tool here to understand the behaviour of the spectral curve, since it can be written in terms of the points of the spectral curve (see (3.27)).

We also make a brief discussion about the invariance of the differential Galois group of the Schrödinger equation under Darboux–Crum transformations.

In the time dependent situation, we study how the classical Darboux–Crum transformations affect the differential polynomials f_j. The key result is Theorem 3.19. This theorem states the structure of the transformed differential polynomials by means of the initial ones.

Finally, we relate the time dependent integrable systems used to construct the KdV hierarchy with the AKNS systems studied in Chapter 2 and used Theorem 2.2 to study the invariance of the differential Galois groups of such integrable systems under Darboux–Crum transformations.

All the results stated in this chapter, unless otherwise indicated, are considered original results and are contained in the preprint [53]. A computational approach to the KdV_1 rational solitons and Darboux transformations is published in our article [52]. They have been presented in the following conferences and seminars:

- XVI Encuentro de Álgebra Computacional y Aplicaciones (EACA 2018), University of Zaragoza, July 2018.
- International Conference on Algebraic Methods in Dynamical Systems (AMDS 2018), Technical University of Madrid, June 2018.
- Seminar of Integrability and Differential Galois Theory, Technical University of Madrid, February 2018.

Chapter 4

In this chapter we compute fundamental matrices for the integrable system (4.1)

$$\begin{cases} \Phi_x & = U\Phi = \begin{pmatrix} 0 & 1 \\ u-E & 0 \end{pmatrix}\Phi, \\ \Phi_{t_r} & = V_r\Phi = \begin{pmatrix} -\dfrac{F_{r,x}(u)}{2} & F_r(u) \\ (u-E)F_r(u) - \dfrac{F_{r,xx}(u)}{2} & \dfrac{F_{r,x}(u)}{2} \end{pmatrix}\Phi, \end{cases}$$

for the rational KdV_r potentials that Adler and Moser construct in [7] and for any value of the energy E. We compute the formulas for fundamental matrices in Theorems 4.7 and 4.13.

The solutions when E is different from zero are given in terms of some functions $Q_n^{\pm}$ which depend on λ, for $\lambda^2 + E = 0$. In Theorem 4.18 we prove that these functions are defined recursively by the same formula as Adler–Moser polynomials and that in fact, they are polynomials in x and in some constants of x, say $\tau_2^{\pm}, \dots, \tau_n^{\pm}$. Thus, these polynomials generalized the Adler–Moser polynomials.

We prove that fundamental matrices present different behaviours depending on whether the energy E is zero or not. Futhermore, there is no specialization process from the non zero energy situation to the zero energy situation, as can be verified in the developed examples.

We link this behaviour with the spectral curve of the associated stationary problem ($t_r = 0$). This curve is $\Gamma : \mu^2 - E^{2n+1} = 0$. We see that $E = 0$ is the only affine singular point of this curve. This stationary situation is reflected in the time dependent case. As a result, the fundamental matrix presents a discontinuity at $E = 0$. See for instance the Theorems 4.10 and 4.15 where their determinants are computed.

Finally, we compute the differential Galois groups of the Picard–Vessiot extensions given by the fundamental matrix for each value of λ and see that, as expected, they are time independent and invariant under Darboux–Crum transformations.

All the results stated in this chapter, unless otherwise indicated, are considered original results and are contained in the preprint [53] and in the article [52]. They have been presented in the following conferences and seminars:

- XVI Encuentro de Álgebra Computacional y Aplicaciones (EACA 2018), University of Zaragoza, July 2018.
- International Conference on Algebraic Methods in Dynamical Systems (AMDS 2018), Technical University of Madrid, June 2018.
- Functional Equations in Limoges (FELIM 2018), XLIM, Limoges (France), March 2018.
- Seminar of Integrability and Differential Galois Theory, Technical University of Madrid, February 2018.
- Seventh Iberoamerican Congress on Geometry, University of Valladolid, January 2018.

Chapter 5

In this chapter we define a λ-parametric family of functions $u_{r,n}^{\pm}$. We show that it can be obtain by means of Darboux–Crum transformations for Adler–Moser potentials and that, in fact, the Adler–Moser potentials are contained in this parametric family.

We prove in Theorem 5.12 that they are indeed KdV_r potentials as well, and we will call them λ-parametric potentials. We also compute fundamental matrices for the Schrödinger equation for these potentials, see Theorems 5.3 and 5.8.

Finally, we study the associated stationary problem for $t_r = 0$ and, using a important result of Duistermaat and Grünbaum (Theorem 5.17), we prove that, for fix n, the stationary potential $u_n^{\pm(0)}$ is a solution of the stationary KdV_n equation, see Proposition 5.18.

Chapter 5 is still an ongoing work. We are writing an article about this.

All the results stated in this chapter, unless otherwise indicated, are considered original and they have been presented in the following conferences and seminars:

- Seminar of Integrability and Differential Galois Theory, Technical University of Madrid, February 2018.

Chapter 6

Finally, in Chapter 6 we study from a galoisian point of view the Darboux transformations for second symmetric power systems and orthogonal differential systems. For this study we will restrict ourselves to a stationary situation with one spacial variable x.

Our starting point is a classical Darboux's Theorem (Theorem 6.3) about Darboux transformations for second order linear differential equations (see [26]). We extend this result to Darboux transformations for second symmetric powers of second order linear differential equations via differential systems, see Theorems 6.13 and 6.14.

Then, performing the gauge change (6.21), we transform these systems into orthogonal differential systems where their coefficient matrices are in the Lie algebra $\mathfrak{so}(3, K)$. As a result, the extended Darboux transformations for second symmetric power systems induce transformations in the orthogonal systems which turns out to be Darboux transformations as well, see Corollaries 6.15 and 6.16.

The main results of this part are Theorems 6.13 and 6.14 and Corollaries 6.15 and 6.16, in which we explicitly compute the corresponding Darboux transformations for second symmetric power systems and orthogonal systems. Then we construct an infinite chain of integrable linear differential systems (in the Picard–Vessiot sense). We also see that the constructed Darboux transformations preserve the differential Galois groups and the eigenrings of the systems.

This approach to the study of orthogonal systems is, as far as we know, different from everything that has been done so far. Until now, the problem of obtaining solutions for orthogonal systems was addressed directly. We propose here an indirect study by means of the tensor product levels associated with the initial system. Moreover, all the results for orthogonal differential systems are obtained as a consequence of the computations and constructions made on the systems of the second symmetric power differential equations.

Chapter 6 is a joint work with P. Acosta-Humánez (from Universidad Simón Bolívar, Barranquilla, Colombia) and with M. Barkatou and J.-A. Weil (from Xlim Research Center, Limoges, France). It is still an ongoing work. We are writing an article about this.

All the results stated in this chapter, unless otherwise indicated, are considered original.

Future works

We end this introduction with a short discussion of some problems we are interested in.

1. We would like to study whether the differential system (2.4)

$$\Sigma_x + [\Sigma, P + J\Sigma] = 0 \qquad \text{and} \qquad \Sigma_t + \sum_{p=0}^{n} [\Sigma, V_p]\Sigma^p = 0\,.$$

   presented in Section 2.2 is a Lie–Vessiot system. A galoisian approach to Lie–Vessiot systems was presented in [13, 14].

2. In Section 4.1.1 we make a galoisian study associated to the spectral curve for $t_r = 0$. It would be really interesting to extend this study to the case $t_r = t_0 \neq 0$, when variation along the flow of the KdV hierarchy may occur. In fact, the study of the *time dependent algebro-geometric initial value problem*

$$L_{t_r} = [A, L], \qquad [L^{(0)}, A^{(0)}_{2n+1}] = 0$$

   in Lax form is an open problem from the computational point of view (see [40] p. 65, where the problem is presented, and then analyzed by means of the theta function representation of the algebro-geometric solutions).

3. We would like to study the differential Galois groups of the integrable system for the Schrödinger equation $(-\partial_{xx} + u^{\pm}_{r,n} - E)\phi = 0$, where $u^{\pm}_{r,n}$ is a λ-potential. Focusing on the stationary case, we also may ask what are the spectral curves for these stationary λ-potentials.

   A really interesting question here is if we can recover all the rational KdV potentials, in both the time dependent and the stationary setting, by means of the potentials $u^{\pm}_{r,n}$ and $u^{\pm(0)}_{n}$ for particular values of λ and $\tau^{\pm}_2, \ldots, \tau^{\pm}_n$.

   We will study these potentials in connection with the work [47].

4. We would like to apply the developed techniques in Chapter 6 to supersymmetric quantum mechanics, Frenet–Serret problem and rigid body problem (see for instance the works [2, 3, 4] where the first of these problems has been treated with a galoisian approach, and [34, 35], where the third of these problems was studied).

Introducción: resumen y conclusiones

Marco histórico

Teoría de Galois Diferencial

En 1883, E. Picard publicó el artículo [73] en el que sentaba las bases de la Teoría de Galois Diferencial, una Teoría de Galois para ecuaciones diferenciales lineales. A este trabajo le siguieron poco tiempo depués los artículos [74, 75, 87], donde E. Picard y su estudiante predoctoral E. Vessiot consolidaron esta nueva teoría. Por esta razón también se llama Teoría de Picard–Vessiot.

Unos años depués, E. Kolchin escribió los trabajos de E. Picard y E. Vessiot en lenguaje moderno y obtuvo nuevos resultados importantes, ver [56, 57, 58]. En concreto, en [58] mostró cómo extender la Teoría de Picard–Vessiot a cuerpos diferenciales con derivadas parciales. I. Kaplanski, en su magnífico libro [54], facilitó la difusión del trabajo de Kolchin.

Más recientemente, en 1986, J. J. Kovacic presentó un algoritmo, el llamado algoritmo de Kovacic, para resolver ecuaciones diferenciales lineales de segundo orden en forma cerrada basada en un teorema de clasificación de subgrupos de $SL(2, \mathbf{C})$, ver [59].

J.-P. Ramis fue el primero en encontrar una conexión entre las singularidades irregulares de ecuaciones diferenciales lineales y la Teoría de Picard–Vessiot. En concreto, demostró que las matrices de Stokes asociadas a este tipo de singularidades pertenecen al grupo de Galois diferencial (ver [20, 66, 77, 78, 79]).

N. Katz y P. Deligne introdujeron la aproximación Tannakiana a la Teoría de Galois Diferencial, ver [31, 55]. En [64], M. Loday estableció el vínculo entre las aproximaciones Tannakiana y analítica a estos problemas de Teoría de Galois Diferencial dadas por los autores anteriores.

En los últimos años se han desarrollado nuevas teorías de Galois diferenciales. Nos gustaría destacar los trabajos de B. Malgrange y H. Umemura, ver [65, 86], en Teoría de Galois Diferencial no lineal; P. J. Cassidy y M. F. Singer, ver [21], en Teoría de Picard–Vessiot Paramétrica; y los trabajos de J. J. Morales-Ruiz y J.-P. Ramis en Teoría de Galois Diferencial para sistemas hamiltonianos, también conocida como Teoría de Morales–Ramis, ver por ejemplo [69, 70].

En esta tesis nos centramos en la Teoría de Picard–Vessiot clásica. Algunas referencias recientes para esta aproximación son [20, 23, 24, 84].

Transformaciones de Darboux

En 1882, G. Darboux presentó en [26] un tipo de transformaciones para ecuaciones diferenciales lineales de segundo orden que permitían obtener soluciones de estas ecuaciones diferenciales (ver también [27, 28]). Unos años después, en 1955, M. Crum describió un método iterativo para aplicar transformaciones de Darboux, ver [25]. Por este motivo, en esta tesis doctoral llamamos a estas transformaciones *transformaciones de Darboux–Crum*.

En los últimos años, muchos autores han estudiado las transformaciones de Darboux desde diferentes puntos de vista, remitimos a [67] y a las referencias allí incluidas. Destacamos también el trabajo de P. Deift, quien hizo un estudio muy general de las transformaciones de Darboux desde un punto de vista espectral, ver [30].

Hay otras dos aproximaciones equivalentes a las transformaciones de Darboux. La primera es la factorización de operadores. En 1941, E. Schrödinger mostró que hay muchas formas de factorizar la ecuación hipergeométrica, ver [81]. Diez años después, L. Infeld y T. E. Hull presentaron en [46] su método de factorización, donde clasificaban las factorizaciones de las ecuaciones diferenciales lineales de segundo orden para problemas de autovalores en mecánica de ondas.

La otra aproximación es en el contexto de la mecánica cuántica supersimétrica. Según el libro de V. B. Matveev y M. A. Salle ([67]), la relación entre la mecánica cuántica supersimétrica y las transformaciones de Darboux fue descubierta por A. Andrianov, N. Borisov y M. Ioffe, ver [9]. Para este enfoque, remitimos a los trabajos originales de E. Witten y L. Gendenshteïn, ver [89] y [39] respectivamente. Y a la tesis doctoral [4] y las referencias allí incluidas, en la que además se presentó una aproximación galoisiana a las transformaciones de Darboux y los potenciales de forma invariante.

En este trabajo nosotros estudiamos la invariancia galoisiana bajo transformaciones de Darboux–Crum.

Ecuaciones en derivadas parciales solitónicas

En 1967, C. Gardner, J. Greene, M. Kruskal y R. Miura encontraron un método para resolver ciertos problemas de valores iniciales relevantes para la ecuación de Korteweg–de Vries, ver [36]. Allí, relacionaron estas soluciones con el problema de scattering inverso para la ecuación de Schrödinger, demostrando que los autovalores de la ecuación de Schrödinger son integrales de la ecuación de Korteweg–de Vries. Este fue el inicio de la Teoría de Solitones.

Un año después, C. Gardner, M. Kruskal y R. Miura publicaron otro artículo ([37]) donde derivaban distintas leyes de conservación y constantes de movimiento de la ecuación de Korteweg–de Vries. Ese mismo año y basándose en su trabajo, P. Lax publicó [60], donde usaba por primera vez el conmutador de dos operadores diferenciales para describir ciertas ecuaciones de evolución, hoy en día conocidos como pares de Lax. Después, P. Lax hizo otras contribuciones relevantes a la ecuación de Korteweg–de Vries, ver [61, 62].

En su trabajo seminal [1], M. J. Ablowitz, D. J. Kaup, A. C. Newell y H. Segur asociaron un sistema diferencial lineal a ciertas familias importantes de ecuaciones de evolución de tipo solitónico, llamados hoy en día sistemas AKNS en su honor.

H. Airault, H. McKean y J. Moser encontraron condiciones para que funciones racionales y elípticas fueran soluciones de la ecuación de Korteweg–de Vries, ver [8]. Algunos años después, M. Adler y J. Moser construyeron en [7] una familia de soluciones racionales de las ecuaciones de Korteweg–de Vries.

J. Duistermaat y F. Grünbaum en [32] hicieron un estudio diferencial del parámetro espectral de la ecuación de Schrödinger y trataron de obtener condiciones para determinar todas las soluciones racionales independientes del tiempo de las ecuaciones de Korteweg–de Vries.

Más recientemente, F. Gesztesy, K. Unterkofler y R. Weikard caracterizaron las tres clases de soluciones algebro-geométricas de la jerarquía KdV (racionales, meromorfas simplemente periódicas y elípticas) por medio de sistemas de Calogero–Moser, ver [41].

Nuestra aproximación al estudio de ecuaciones solitónicas se enmarca dentro de las aproximaciones citadas anteriormente.

Estado del arte

A lo largo de los últimos años se han publicado muchos trabajos en los que se relacionan algunos de los temas anteriormente tratados. Yu.V. Brezhnev aplicó la Teoría de Picard–Vessiot a las soluciones algebro-geométricas (o de zona finita) de las ecuaciones solitónicas, ver [16, 17, 18]. En [4] se dio una aproximación galoisiana a las transformaciones de Darboux en mecánica cuántica supersimétrica unidimensional, ver también [2, 3].

E. Horozov, usando el trabajo de J. Duistermaat y F. Grünbaum, estudió en [44] el problema biespectral y describió todos los operadores biespectrales contenidos en el álgebra de Weyl y sus transformaciones de Darboux, las llamadas transformaciones de Darboux biespectrales.

Recientemente, la Teoría de Galois Diferencial se ha aplicado de una manera muy original al estudio de la integrabilidad de campos vectoriales polinomiales en el plano, ver [5, 6], y al problema de encontrar soluciones en forma cerrada de la ecuación de

Schrödinger para potenciales de KdV estacionarios, ver [71]. En este último trabajo, los autores también estudiaron por primera vez las extensiones diferenciales dadas por estas soluciones en función del parámetro espectral, la Teoría de Picard–Vessiot Espectral. En esta tesis nosotros no tratamos con extensiones espectrales, pero serán clave para entender el comportamiento de las extensiones diferenciales que podemos obtener sin fijar el parámetro espectral.

En el caso unidimensional, la escuela de Kyoto ha estudiado las ecuaciones diferenciales lineales en el parámetro espectral y han desarrollado una teoría general de deformaciones que preservan la monodromía para sistemas de ecuaciones diferenciales lineales ordinarias. Ver los trabajos de M. Jimbo, T. Miwa y K. Ueno [48, 49, 50].

Este año T. Combot en [22] ha dado una clasificación completa de potenciales racionales integrables para la ecuación de Schrödinger unidimensional con parámetro espectral.

Nuestro trabajo sobre la ecuación de Schrödinger se enmarca en el caso $1+1$ dimensional, con especial interés en los potenciales racionales.

Estructura y contenidos

Esta tesis doctoral está estructurada en seis capítulos. El hilo conductor de todos ellos son la invariancia galoisiana bajo transformaciones de Darboux y la invariancia del tiempo de los grupos de Galois diferenciales de los sistemas diferenciales estudiados. A continuación describiremos brevemente los contenidos de cada uno, destacando en cada caso los resultados originales principales.

Capítulo 1

A lo largo del primer capítulo, daremos los conceptos y resultados preliminares relativos a sistemas integrables, operadores diferenciales, Teoría de Galois Diferencial y la ecuación de Schrödinger necesarios para desarrollar nuestro trabajo, el cual será expuesto en los siguientes cinco capítulos. En este capítulo no hay resultados originales.

Capítulo 2

El segundo capítulo está dedicado a estudiar cómo se comportan los grupos de Galois diferenciales de los sistemas lineales de tipo (Zhakharov–Shabat) AKNS bajo transformaciones de Darboux matriciales. Un sistems AKNS es un sistema integrable de ecuaciones en derivadas parciales, introducido y llamado así por M. J. Ablowitz, D. J. Kaup, A. C. Newell y H. Segur (AKNS) en 1974 (ver [1]). Las condiciones de curvatura nula de estos sistemas son ecuaciones en derivadas parciales no lineales. Por ejemplo, incluyen todas las jerarquías solitónicas clásicas como la jerarquía KdV, una secuencia infinita de ecuaciones en derivadas parciales que empieza con la ecuación de Korteweg–de Vries.

Una transformación de Darboux para un sistema AKNS es una transformación gauge que es covariante con respecto a la estructura AKNS. Estamos especialmente interesados en la extensión diferencial que las transformaciones de Darboux producen en el cuerpo de coeficientes del sistema AKNS.

El resultado principal de esta parte es el Teorema 2.2, que afirma que el grupo de Galois diferencial del sistema transformado por medio de una transformación de Darboux es isomorfo a un subgrupo del grupo de Galois diferencial del sistema inicial. Por tanto, desde nuestro punto de vista esto significa que el sistema transformado es al menos tan integrable (en el sentido de Picard–Vessiot) como el sistema inicial. En particular, si el sistema inicial es integrable, el sistema transformado es integrable también, en completo acuerdo con las ideas de Darboux (ver [28, p. 10]).

Obtendremos el isomorfismo entre las componentes conexas de la identidad de ambos grupos de Galois diferenciales cuando el cuerpo de coeficientes del sistema transformado sea una extensión algebraica del cuerpo de coeficientes del sistema inicial (ver Corolario 2.5).

Hasta donde nosotros sabemos, esta es la primera vez que los sistemas AKNS se han tratado desde un punto de vista galoisiano. Todos los resultados enunciados en este capítulo, salvo indicado otro caso, se consideran resultados originales.

Los resultados contenidos en este capítulo han sido publicados en nuestro artículo [51] y han sido presentados en los siguientes congresos y seminarios:

- International Conference on Algebraic Methods in Dynamical Systems (AMDS 2018), Universidad Politécnica de Madrid, Junio 2018.

- Functional Equations in Limoges (FELIM 2018), Centro de investigación XLIM, Limoges (Francia), Marzo 2018.

- Seminario de Cálculo Formal, XLIM, Limoges (Francia), Marzo 2018.

- Finite Dimensional Integrable Systems in Geometry and Mathematical Physics (FDIS 2017), Centre de Recerca Matemàtica (CRM), Barcelona, Julio 2017.

- Geometry of Singularities and Differential Equations (GSDS 2017), Universidad de Cantabria, Junio 2017.

- Seminario de Integrabilidad y Teoría de Galois Diferencial, Universidad Politécnica de Madrid, Abril 2016.

Capítulo 3

Durante el Capítulo 3 desarrollamos la jerarquía KdV tanto para el caso estacionario como para el caso temporal (caso $1+1$ dimensional). Estas ecuaciones diferenciales no lineales se obtienen como condiciones de curvatura nula de una familia de sistemas integrables en función de la derivada espacial, pongamos ∂_x, de ciertos polinomios diferenciales f_j definidos recursivamente (ver (3.1)).

Para potenciales estacionarios, consideramos la curva espectral Γ asociada al operador de Schrödinger

$$\mathcal{L} - E = -\partial_{xx} + u - E$$

para cada nivel de la jerarquía KdV estacionario. Normalmente este curva se considera como una curva algebraica plana y puede ser una curva singular plana. En nuestra aproximación a las transformaciones de Darboux para este tipo de sistemas hemos tenido que considerar Γ pero en $\mathbb{P}^2$, el plano proyectivo complejo. Así incluimos en nuestro análisis el punto del infinito de estas curvas.

Estudiamos cómo cambia Γ cuando realizamos una transformación de Darboux–Crum clásica al operador de Schrödinger. La acción de la transformación depende fuertemente del valor de E que elegimos para calcularla. El Teorema 3.16 establece este comportamiento. Encontramos que, cuando el valor de E corresponde a un punto regular de la curva, la curva espectral permanece invariante. En otro caso, la nueva curva es un blowing-up o blowing-down de la curva inicial dependiendo del tipo de transformación de Darboux realizada.

La diagonal de la función de Green será una herramienta esencial para entender el comportamiento de la curva espectral, ya que se puede escribir en función de los puntos de la curva espectral (ver (3.27)).

Además damos una breve explicación sobre la invariancia del grupo de Galois diferencial de la ecuación de Schrödinger bajo transformaciones de Darboux.

En la situación temporal, estudiamos cómo las transformaciones de Darboux–Crum clásicas afectan a los polinomios diferenciales f_j. El resultado clave es el Teorema 3.19. Este teorema establece la estructura de los polinomios diferenciales transformados en función de los iniciales.

Finalmente, relacionamos los sistemas integrables temporales utilizados para construir la jerarquía KdV con los sistemas AKNS estudiados en el Capítulo 2 y usamos el Teorema 2.2 para estudiar la invariancia de los grupos de Galois diferenciales de dichos sistemas integrables bajo transformaciones de Darboux–Crum.

Todos los resultados enunciados en este capítulo, salvo indicado otro caso, se consideran resultados originales y están contenidos en el preprint [53]. Una aproximación computacional a los solitones racionales de KdV_1 y a las transformaciones de Darboux está publicada en nuestro artículo [52]. Han sido presentados en los siguientes congresos y seminarios:

- XVI Encuentro de Álgebra Computacional y Aplicaciones (EACA 2018), Universidad de Zaragoza, Julio 2018.

- International Conference on Algebraic Methods in Dynamical Systems (AMDS 2018), Universidad Politécnica de Madrid, Junio 2018.

- Seminario de Integrabilidad y Teoría de Galois Diferencial, Universidad Politécnica de Madrid, Febrero 2018.

Capítulo 4

En este capítulo calculamos matrices fundamentales para el sistema integrable (4.1)

$$\begin{cases} \Phi_x & = U\Phi = \begin{pmatrix} 0 & 1 \\ u-E & 0 \end{pmatrix} \Phi, \\ \Phi_{t_r} & = V_r\Phi = \begin{pmatrix} -\dfrac{F_{r,x}(u)}{2} & F_r(u) \\ (u-E)F_r(u) - \dfrac{F_{r,xx}(u)}{2} & \dfrac{F_{r,x}(u)}{2} \end{pmatrix} \Phi, \end{cases}$$

para los potenciales racionales de KdV_r que Adler y Moser construyeron en [7] y para cualquier valor de la energía E. Calculamos las fórmulas de las matrices fundamentales en los Teoremas 4.7 y 4.13.

Las soluciones cuando E es diferente de cero vienen dadas en función de ciertas funciones $Q_n^{\pm}$ que dependen de λ, para $\lambda^2 + E = 0$. En el Teorema 4.18 probamos que estas funciones están definidas recursivamente por la misma fórmula que los polinomios de Adler–Moser y que, de hecho, son polinomios en x y en ciertas constantes de x, pongamos $\tau_2^{\pm}, \dots, \tau_n^{\pm}$. Por tanto, estos polinomios generalizan los polinomios de Adler–Moser.

Probamos que las matrices fundamentales presentan comportamientos diferentes dependiendo de si la energía E es cero o no. Es más, no hay proceso de especialización desde la situación de energía no cero hasta la situación de energía cero, como podemos verificar en los ejemplos desarrollados.

Relacionamos este comportamiento con la curva espectral del problema estacionario asociado ($t_r = 0$). Esta curva es $\Gamma : \mu^2 - E^{2n+1} = 0$. Vemos que $E = 0$ es el único punto singular afín de esta curva. Esta situación estacionaria se refleja en el caso temporal. Como resultado, la matriz fundamental presenta una discontinuidad en $E = 0$. Ver, por ejemplo, los Teoremas 4.10 y 4.15 donde calculamos sus determinantes.

Finalmente, calculamos los grupos de Galois diferenciales de las extensiones de Picard–Vessiot dadas por la matriz fundamental para cada valor de λ y vemos que, como esperábamos, son independientes del tiempo e invariantes bajo transformaciones de Darboux–Crum.

Todos los resultados enunciados en este capítulo, salvo indicado otro caso, se consideran resultados originales y están contenidos en el preprint [53] y en el artículo [52]. Han sido presentados en los siguientes congresos y seminarios:

- XVI Encuentro de Álgebra Computacional y Aplicaciones (EACA 2018), Universidad de Zaragoza, Julio 2018.
- International Conference on Algebraic Methods in Dynamical Systems (AMDS 2018), Universidad Politécnica de Madrid, Junio 2018.
- Functional Equations in Limoges (FELIM 2018), XLIM, Limoges (France), Marzo 2018.

- Seminario de Integrabilidad y Teoría de Galois Diferencial, Universidad Politécnica de Madrid, Febrero 2018.

- Seventh Iberoamerican Congress on Geometry, Universidad de Valladolid, Enero 2018.

Capítulo 5

En este capítulo definimos una familia λ-paramétrica de funciones $u^{\pm}_{r,n}$. Mostramos que se puede obtener mediante transformaciones de Darboux–Crum de los potenciales de Adler–Moser y que, de hecho, los potenciales de Adler–Moser están contenidos en esta familia paramétrica.

En el Teorema 5.12 probamos que en efecto son potenciales de KdV_r, y los llamaremos potenciales λ-paramétricos. Calculamos además matrices fundamentales para la ecuación de Schrödinger para estos potenciales, ver los Teoremas 5.3 y 5.8.

Finalmente, estudiamos el problema estacionario asociado para $t_r = 0$ y, utilizando un importante resultado de Duistermaat y Grünbaum (Teorema 5.17), probamos que, para n fijado, el potencial estacionario $u_n^{\pm(0)}$ es una solución de la ecuación KdV_n estacionaria, ver la Proposición 5.18.

El Capítulo 5 es aún un trabajo en desarrollo. Estamos escribiendo un artículo sobre ello.

Todos los resultados enunciados en este capítulo, salvo indicado otro caso, se consideran resultados originales y han sido presentados en los siguientes congresos y seminarios:

- Seminario de Integrabilidad y Teoría de Galois Diferencial, Universidad Politécnica de Madrid, Febrero 2018.

Capítulo 6

Finalmente, en el Capítulo 6 estudiamos desde un punto de vista galoisiano las transformaciones de Darboux para sistemas asociados a segundas potencias simétricas y sistemas diferenciales ortogonales. Para este estudio nos restringimos a una situación estacionaria con una variable espacial x.

Nuestro punto de partida es un teorema clásico de Darboux (Teorema 6.3) sobre transformaciones de Darboux para ecuaciones diferenciales lineales de segundo orden (ver [26]). Extendemos este resultado a transformaciones de Darboux para segundas potencias simétricas de ecuaciones diferenciales de segundo orden vía sistemas diferenciales, ver Teoremas 6.13 y 6.14.

Entonces, realizando la transformación gauge (6.21), transformamos estos sistemas en sistemas diferenciales ortogonales donde sus matrices de coeficientes están en el álgebra de Lie $\mathfrak{so}(3, K)$. Como resultado, las transformaciones de Darboux extendidas para sistemas de segundas potencias simétricas inducen transformaciones en los sistemas ortogonales que resultan ser transformaciones de Darboux también, ver Corolarios 6.15 y 6.16.

Los resultados principales de esta parte son los Teoremas 6.13 y 6.14 y los Corolarios 6.15 y 6.16, en los que calculamos explícitamente las transformaciones de Darboux correspondientes a sistemas para segundas potencias simétricas y sistemas ortogonales. Entonces construimos una cadena infinita de sistemas lineales integrables (en el sentido de Picard–Vessiot). Además, vemos que las transformaciones de Darboux construidas preservan los grupos de Galois diferenciales y los «eigenrings» de los sistemas.

Esta aproximación al estudio de sistemas ortogonales es, hasta donde nosotros sabemos, diferente de todo lo que se ha hecho hasta la fecha. Hasta ahora, el problema de obtener soluciones para sistemas ortogonales se abordaba directamente. Nosotros proponemos aquí un estudio indirecto por medio de las capas tensoriales asociadas con el sistema inicial. Además, todos los resultados para sistemas diferenciales ortogonales se obtienen como consecuencia de los cálculos y las construcciones hechas para las ecuaciones diferenciales para segundas potencias simétricas.

El Capítulo 6 es un trabajo conjunto con P. Acosta-Humánez (de la Universidad Simón Bolívar, Barranquilla, Colombia) y con M. Barkatou y J.-A. Weil (del Centro de Investigación Xlim, Limoges, Francia). Es aún un trabajo en desarrollo. Estamos escribiendo un artículo sobre ello.

Todos los resultados enunciados en este capítulo, salvo indicado otro caso, se consideran resultados originales.

Trabajos futuros

Terminamos esta introducción con una breve discusión de algunos problemas en los que estamos interesados.

1. Nos gustaría estudiar si el sistema diferencial (2.4)

$$\Sigma_x + [\Sigma, P + J\Sigma] = 0 \qquad \text{y} \qquad \Sigma_t + \sum_{p=0}^{n} [\Sigma, V_p]\Sigma^p = 0 \,.$$

   presentado en la Sección 2.2 es un sistema de Lie–Vessiot. Una aproximación galoisiana a los sistemas de Lie–Vessiot se presentó en [13, 14].

2. En la Sección 4.1.1 hacemos un estudio galoisiano asociado a la curva espectral para $t_r = 0$. Sería muy interesante extender dicho estudio al caso $t_r = t_0 \neq 0$, donde podríamos tener variación a lo largo del flujo de la jerarquía KdV. De hecho, el estudio del *problema temporal algebro-geométrico de valor inicial*

$$L_{t_r} = [A, L], \qquad [L^{(0)}, A^{(0)}_{2n+1}] = 0$$

   en forma de Lax es un problema abierto desde el punto de vista computacional (ver [40] p. 65, donde se presenta el problema, y después se analiza por medio de la representación en funciones theta de las soluciones algebro-geométricas).

3. Nos gustaría estudiar los grupos de Galois diferenciales de los sistemas integrables para la ecuación de Schrödinger $(-\partial_{xx}+u_{r,n}^{\pm}-E)\phi=0$, donde $u_{r,n}^{\pm}$ es un λ-potencial. Centrándonos en el caso estacionario, también nos preguntamos cuáles son las curvas espectrales para estos λ-potenciales estacionarios.

   Una pregunta realmente interesante aquí es si podemos recuperar todos los potenciales de KdV racionales, tanto en el caso temporal como en el estacionario, por medio de los potenciales $u_{r,n}^{\pm}$ y $u_n^{\pm(0)}$ para valores particulares de λ y $\tau_2^{\pm},\ldots,\tau_n^{\pm}$.

   Estudiaremos estos potenciales en conexión con el trabajo [47].

4. Nos gustaría aplicar las técnicas desarrolladas en el Capítulo 6 a la mecánica cuántica supersimétrica, al problema de Frenet–Serret y al problema del cuerpo rígido (ver, por ejemplo, los trabajos [2, 3, 4], donde se trata el primero de estos problemas con un enfoque galoisiano, y [34, 35], donde se estudia el tercero de estos problemas).

Chapter 1

Preliminaries

In this chapter we expose some basic concepts and results concerning integrable systems, differential operators, differential Galois Theory and the Schrödinger equation that will be necessary along this thesis.

1.1 Integrable systems

For this section we follow closely [84], Chapter 1 and Appendix D. Let K be a differential field with commuting derivations $\partial_{x_1}, \partial_{x_2}, \dots, \partial_{x_n}$ and field of constants $\mathbf{C} = \{c \in K : \partial_{x_i} c = 0 \text{ for } i = 1, \dots, n\}$ algebraically closed and of characteristic zero. Consider the system of matrix partial differential equations over K:

$$\partial_{x_i}\Phi = A_i\Phi, \qquad A_i \in \mathrm{Mat}_m(K),\ i = 1, \dots, n. \tag{1.1}$$

Let Φ be a solution of the system. The fact that the derivatives of Φ commute translate into the following compatibility conditions in the matrix coefficients A_i:

$$\partial_{x_j}A_i - \partial_{x_i}A_j + [A_i, A_j] = 0, \qquad \text{for all } 1 \leq i, j \leq n. \tag{1.2}$$

These relations are the so-called *zero curvature conditions of system* (1.1). When differential system (1.1) satisfies the zero curvature conditions we say that it is an *integrable system*.

Let L be a differential extension of K. A fundamental matrix for system (1.1) is an invertible matrix $\Phi \in GL_m(L)$ such that $\partial_{x_i}\Phi = A_i\Phi$ holds for $i = 1, \dots, n$. The set of all fundamental matrices for system (1.1) is $\Phi \cdot GL_m(\mathbf{C})$.

We have the following result for fundamental matrices of integrable systems:

Theorem 1.1 (Liouville's Formula)**.** *Let Φ be a fundamental matrix for system* (1.1)*. Then the determinant of Φ satisfies*

$$\partial_{x_i}(\det\Phi) = \operatorname{tr} A_i \det\Phi, \qquad i = 1, \dots, n.$$

1.1.1 Picard–Vessiot extensions

Let Φ be a fundamental solution of (1.1). A *Picard–Vessiot ring* over K for system (1.1) is a simple differential ring R over K such that $\Phi \in GL_m(R)$ and it is generated as a ring by K, the entries of the fundamental matrix Φ and the inverse of its determinant.

We define a *Picard–Vessiot field* L over K for system (1.1) as the field of fractions of a Picard–Vessiot ring for such system.

The following result ensures us that we can always find a Picard–Vessiot extension $K \to L$ for the system and such extension does not enlarge the field of constants of K.

Proposition 1.2. *Consider system* (1.1)*, the following statements are satisfied:*

1. *There exists a Picard–Vessiot ring for the system,*
2. *Any two Picard–Vessiot rings for the system are isomorphic, and*
3. *The field of constants of the Picard–Vessiot field is again* $\mathbf{C}$*.*

In other words, a Picard–Vessiot field for system (1.1) is the smallest differential extension of K containing the solutions of system (1.1) and having the same field of constants as K.

We end this section with a characterization of the Picard–Vessiot field for system (1.1):

Proposition 1.3. *The field L is a Picard–Vessiot field for system* (1.1) *if and only if the following conditions hold:*

1. *The field of constants of L is* $\mathbf{C}$*,*
2. *There exists a fundamental matrix $\Phi \in GL_m(L)$ for the system, and*
3. *L is generated over K by the entries of Φ.*

1.1.2 Differential Galois Theory for matrix differential systems

Next, we briefly develop the differential Galois Theory for matrix differential systems, also known as Picard–Vessiot Theory.

The differential automorphisms of a Picard–Vessiot field L for the system are the automorphisms that commutes with all the derivations of L. The *differential Galois group* of system (1.1) is the group G of differential automorphisms of L leaving invariant the elements of K, in other words,

$$G = \{\sigma : L \to L, \text{ such that } \sigma(f) = f \text{ if } f \in K \text{ and } \sigma(\partial_{x_i}) = \partial_{x_i}(\sigma),\ i = 1, \ldots, n\}.$$

Differential Galois group as group of matrices. The differential Galois group can be represented as a subgroup of $GL_m(\mathbf{C})$ as follows. Let Φ be a fundamental matrix for system (1.1) and $\sigma \in G$. The matrix $\sigma(\Phi)$ is also a fundamental matrix for the system, thus, $\sigma(\Phi) = \Phi \cdot C(\sigma)$, where $C(\sigma) \in GL_m(\mathbf{C})$. Therefore, the map

$$G \to GL_m(\mathbf{C}) \text{ given by } \sigma \mapsto C(\sigma)$$

is an injective group homomorphisms.

Hereafter in this Ph.D. thesis we will refer to the differential Galois group simply as Galois group.

Theorem 1.4. *Let L be the Picard–Vessiot field of system* (1.1) *and G its Galois group.*

1. *G considered as subgroup of $GL_m(\mathbf{C})$ is an algebraic group.*
2. *The Lie algebra of G coincides with the Lie algebra of the derivations L/K that commutes with the derivations on L.*
3. *The field L^G of G-invariant elements of L is equal to K.*

An algebraic group G has a unique connected normal algebraic subgroup G^0 of finite index. This means that the connected identity component G^0 is the largest connected algebraic subgroup of G containing the identity. When $G = G^0$, then G is a connected group.

We say that an algebraic group G virtually satisfies some property when its connected identity component G^0 satisfies that property. In this way, we say that system (1.1) is *integrable in the Picard–Vessiot sense* whether its Galois group G is *virtually solvable*, in other words, whether G^0 is solvable. Recall that a group G is solvable if and only if there exists a chain of normal subgroups

$$\{Id\} = G_0 \lhd G_1 \lhd \cdots \lhd G_n = G$$

such that the quotient G_i/G_{i-1} is an abelian group, for $j = 1, \ldots n$.

Remark 1.5. We see then that in this thesis two different notions of integrability for differential system (1.1) will coexist. In order to distinguish them, on the one hand, we will refer to *integrability* or *flatness* when system (1.1) satisfies the zero curvature conditions (1.2). And, on the other hand, we will refer to *Picard–Vessiot integrability* or *integrability in the Picard–Vessiot sense* when the Galois group of system (1.1) is virtually solvable.

Next, we give a geometric interpretation of the connection between the Picard–Vessiot rings and the action of the Galois group.

Let X be an affine variety over K. We denote by $\mathcal{O}(X)$ its coordinate ring. Then, we define the set of $\overline{K}$–points of X as $X(\overline{K}) = \{z : \mathcal{O}(X) \to \overline{K}$, such that $z(f) = f$ if $f \in K\}$.

Let G be the Galois group of system (1.1). Denote $G_K = G \otimes_{\mathbf{C}} K$. Then, a principal homogeneous space over G, also called *G-torsor*, Z is an affine variety over K with a G-action, i.e., a morphism $G_K \times_K Z \to Z$ given by $(g, z) \mapsto z \cdot g$, such that:

1. For all $z \in Z(\overline{K})$, $g_1, g_2 \in G(\overline{K})$, we have $z \cdot 1 = z$ and $z \cdot (g_1 g_2) = (z \cdot g_1) \cdot g_2$.
2. The morphism $G_K \times_K Z \to Z \times_K Z$, given by $(g, z) \mapsto (z \cdot g, z)$, is an isomorphism.

We denote by $Z = \mathrm{Max}(R)$ the maximal spectrum of a Picard–Vessiot ring R.

Theorem 1.6. *Let R be a Picard–Vessiot ring for system* (1.1) *with Galois group G. Then, $Z := \mathrm{Max}(R)$ is a G-torsor over K.*

We end this section by establishing the differential Galois correspondence, analogous to the classical Galois correspondence.

Theorem 1.7 (The Galois Correspondence). *Let $\partial_{x_i}\Phi = A_i\Phi$, $i = 1, \ldots, n$, be a system of matrix partial differential equations over K with Picard–Vessiot field L and Galois group G. Consider the two sets*

$$\mathcal{S} := \{\textit{closed subgroups of } G\},$$
$$\mathcal{L} := \{\textit{differential subfields } M \textit{ of } L, \textit{ containing } K\}.$$

Define the maps:

$$\alpha : \mathcal{S} \to \mathcal{L}, \ \textit{given by } H \mapsto L^H \quad \textit{and} \quad \beta : \mathcal{L} \to \mathcal{S}, \ \textit{given by } M \mapsto \mathrm{Gal}(L/M),$$

where L^H is the subfield of L consisting of the H-invariant elements and $\mathrm{Gal}(L/M)$ is the subgroup of G consisting of the M-linear differential automorphisms. Then,

1. *The maps α and β are inverses of each other.*
2. *The subgroup $H \in \mathcal{S}$ is a normal subgroup of G if and only if $M = L^H$ is, as a set, invariant under G. If $H \in \mathcal{S}$ is normal then the canonical map $G \to \mathrm{Gal}(M/K)$ is surjective and has kernel H. Moreover, M is a Picard–Vessiot field for some linear differential equation over K.*
3. *Let G^0 denote the identity component of G. Then $L^{G^0} \supset K$ is a finite Galois extension with Galois group G/G^0 and it is the algebraic closure of K in L.*

1.2 Differential operators and matrix differential equations

In this section we briefly introduce the Differential Galois Theory for linear differential equations in one variable. We follow [2, 3, 10, 15, 54, 84].

Let K be a differential field in the variable x with derivation $\partial_x = {}'$ and with field of constants $\mathbf{C} = \{c \in K : c' = 0\}$ algebraically closed and of characteristic zero. Take the noncommutative ring $K[\partial]$ consisting of all the expressions of the form $a_n\partial^n + a_{n-1}\partial^{n-1} + \cdots + a_1\partial + a_0$, with $a_i \in K$ for $i = 0, 1, \ldots, n$. The multiplication in this ring is given by $[\partial, a] = a'$. This ring is called *ring of differential operators*.

Let $\mathcal{P} = \partial^n + a_{n-1}\partial^{n-1} + \cdots + a_1\partial + a_0$ be a differential operator in $K[\partial]$, we denote the action of $\mathcal{P}$ on an element y in K or in a differential extension of K by $\mathcal{P}(y) = \partial^n(y) + a_{n-1}\partial^{n-1}(y) + \cdots + a_1\partial(y) + a_0 y$, where $\partial(y) = y'$. So, setting $\mathcal{P}(y) = 0$ we obtain the linear differential equation

$$\mathcal{P}(y) = \partial^n(y) + a_{n-1}\partial^{n-1}(y) + \cdots + a_1\partial(y) + a_0 y = 0. \tag{1.3}$$

There exists a simple way of obtaining a matrix differential equation from a linear differential equation of this form. For this, we consider the *companion matrix* $A_{\mathcal{P}}$ of the differential operator $\mathcal{P}$. This matrix is:

$$A_{\mathcal{P}} = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & \cdots & -a_{n-1} \end{pmatrix}.$$

Then, if y is a solution of equation (1.3), we get that $Y := (y, y', \ldots, y^{(n-1)})^t$ is a column solution of the matrix differential equation:

$$\Phi' = A_{\mathcal{P}}\Phi. \tag{1.4}$$

Let $\langle y_1, y_2, \ldots, y_n\rangle$ be a basis of solutions of $\mathcal{P}(y) = 0$, hence, from the above it is immediate that the wronskian matrix of $y_1, y_2, \ldots, y_n$,

$$W(y_1, y_2, \ldots, y_n) = \begin{pmatrix} y_1 & y_2 & \cdots & y_n \\ y_1' & y_2' & \cdots & y_n' \\ \vdots & \vdots & \cdots & \vdots \\ y_1^{(n-1)} & y_2^{(n-1)} & \cdots & y_n^{(n-1)} \end{pmatrix}$$

is a fundamental matrix for equation (1.4).

We denote by $w(y_1, y_2, \ldots, y_n) = \det W(y_1, y_2, \ldots, y_n)$ the wronskian of $y_1, \ldots, y_n$. Then, we have the following result.

Lemma 1.8. *The wronskian $w(y_1, y_2, \ldots, y_n)$ satisfies the differential equation:*

$$w' = -a_{n-1}w. \tag{1.5}$$

1.2.1 Picard–Vessiot extensions and differential Galois theory

Assume $\langle y_1, y_2, \ldots, y_n\rangle$ is a basis of solutions of $\mathcal{P}(y) = 0$ and $L = K(y_1, y_2, \ldots, y_n, y_1', y_2', \ldots, y_n', \ldots, y_1^{(n-1)}, y_2^{(n-1)}, \ldots, y_n^{(n-1)})$ is the smallest differential extension of K containing the solutions of $\mathcal{P}(y) = 0$ and having the same field of constants of K. The differential extension L is called the *Picard–Vessiot extension* of $\mathcal{P}(y) = 0$.

The differential automorphisms of L are the automorphisms that commutes with the derivation. The differential Galois group of equation $\mathcal{P}(y) = 0$, denoted by G, is the group of differential automorphisms of L leaving invariant the elements of K, in other words,

$$G = \{\sigma : L \to L, \text{ such that } \sigma(f) = f \text{ if } f \in K \text{ and } \sigma(\partial) = \partial(\sigma)\}.$$

All the results exposed in subsections 1.1.1 and 1.1.2 remain true for Picard–Vessiot extensions and Galois groups of linear differential equations. Therefore, we find that $G \subseteq GL_m(\mathbf{C})$ is an algebraic group.

Recall that the wronskian $w(y_1, y_2, \ldots, y_n)$ satisfies differential equation (1.5). We have the following results for the Galois group of equation $\mathcal{P}(y) = 0$.

Lemma 1.9 (Kaplansky, [54]). *$G \subseteq SL_m(\mathbf{C})$ if and only if there exists $p \in K$ such that $a_{n-1} = (\log p)'$. In particular, if $a_{n-1} = (\log p)'$, then $w(y_1, y_2, \ldots, y_n) \in K$.*

Let $V(\mathcal{P})$ be the solution space of equation $\mathcal{P}(y) = 0$. We say that the action of G on $V(\mathcal{P})$ is either

1. *reducible*, if there exists a non-trivial subspace $W \subset V(\mathcal{P})$ such that $G(W) \subset W$. We say that G is *irreducible* if G is not reducible.

2. *imprimitive*, if G is irreducible and there exists subspaces V_i such that $V = V_1 \otimes \cdots \otimes V_m$, where G permutes transitively the V_i, i.e, for all $i = 1, ..., m$ and for all $\sigma \in G$, there exists $j \in \{1, ..., m\}$ such that $\sigma(V_i) = V_j$.
   We say that G is *primitive* if G is irreducible and not imprimitive.

We denote by G^0 the identity connected component of G. As in previous section, we say that G is virtually solvable if G^0 is solvable. Then, differential equation $\mathcal{P}(y) = 0$ is *integrable in the Picard–Vessiot sense* if and only if its Galois group G is *virtually solvable*. We find the following theorem:

Theorem 1.10 (Lie–Kolchin). *G is virtually solvable if and only if G^0 is triangularizable, that is, conjugate to a triangular group.*

We define L to be a *Liouvillian extension* of K if there exists a chain of differential fields $K = L_0 \subset L_1 \subset \cdots \subset L_m = L$ such that $L_i = L_{i-1}(\eta_i)$, for $i = 1, \ldots, m$, where either

1. η_i is algebraic over L_{i-1}, or

2. η_i is primitive or integral over L_{i-1}, i.e., $\eta_i' \in L_{i-1}$, or

3. η_i is exponential over L_{i-1}, i.e., $\eta_i'/\eta \in L_{i-1}$.

The following result shows that Liouvillian extensions characterize the integrability of linear differential equations:

Theorem 1.11 (Kolchin). *Let L be a Picard–Vessiot field of equation* (1.3) *and G its Galois group. The following statements are equivalent:*

1. *G^0 is a solvable group.*

2. *L is a Liouvillian extension of K.*

3. *L is contained in a Liouvillian extension of K.*

Therefore, we will have that the differential equation $\mathcal{P}(y) = 0$ is integrable in the Picard–Vessiot sense if and only if its Picard–Vessiot field L is a Liouvillian extension of K.

It also may occur that only some of the solutions of the differential equation $\mathcal{P}(y) = 0$ are Liouvillian solutions. This result considers this situation.

Proposition 1.12. *Let $\mathcal{P}(y) = 0$ be a differential equation with coefficients in K and Picard–Vessiot field L. Suppose that $\mathcal{P}(y) = 0$ has a nonzero solution in some Liouvillian extension of K. Then, there is a nonzero solution $y \in K$ of $\mathcal{P}(y) = 0$ such that y'/y is algebraic over K.*

1.2.2 Differential Galois theory and eigenrings of differential equations of order two

In this subsection we restrict ourselves to linear differential equations of order two. Consider the second order linear differential equation

$$\mathcal{P}(y) = y'' + py' + qy = 0, \qquad p, q \in K. \tag{1.6}$$

Assume $\langle y_1, y_2 \rangle$ is a basis of solutions of $\mathcal{P}(y) = 0$ and L is the Picard–Vessiot extension of $\mathcal{P}(y) = 0$. We denote by $V(\mathcal{P})$ the solution space of $\mathcal{P}(y) = 0$ and by G the Galois group of this equation.

Remark 1.13. Assume $p = 0$ in (1.6), i.e., $\mathcal{P}(y) = y'' + qy = 0$. Given a particular solution y_1 of this differential equation, we can compute a second independent solution as

$$y_2 = y_1 \int \frac{dx}{y_1^2}\,.$$

This standard procedure is called d'Alembert reduction.

The change of variable

$$\nu := \frac{y'}{y}$$

transforms equation (1.6) into the Riccati equation

$$\nu' = -q - p\nu - \nu^2. \tag{1.7}$$

This is a classical method to obtain a Riccati equation from a second order linear differential equation and vice versa and it can be stated more generally as (see, for instance, [45])

Lemma 1.14 (Ince, [45])**.** *Given functions $a, b, c \in K$, we consider the Riccati equation*

$$u' = a + bu + cu^2.$$

The change of variables

$$u = -\frac{1}{c} \cdot \frac{y'}{y}$$

transforms the Riccati equation into the second order linear differential equation

$$y'' - \left(b + \frac{c'}{c} \right) y' + acy = 0.$$

Conversely, given the second order linear differential equation

$$y'' + \widetilde{a} y' + \widetilde{b} y = 0,$$

the change of variables

$$u = \frac{y'}{y}$$

transforms it into the Riccati equation

$$u' + u^2 + \widetilde{a} u + \widetilde{b} = 0.$$

The integrability of equation $\mathcal{P}(y) = 0$ is characterize in terms of the algebraic solutions of the Riccati equation (1.7), as the following theorem shows.

Theorem 1.15 (Liouville, [63])**.** *The differential equation* (1.6) *is integrable if and only if the associated Riccati equation* (1.7) *has an algebraic solution over the field* K*.*

Moreover, Kovacic in his article [59] develop an algorithm to solve the equation $\mathcal{P}(y) = 0$ in closed form using the solutions of the associated Riccati equation. This algorithm is based on the following theorem.

Theorem 1.16. *Let G be an algebraic subgroup of $SL(2, \mathbf{C})$. Then one of the following four cases can occur.*

1. *G is triangularizable and there exists a rational solution of the Riccati equation* (1.7)*.*

2. *G is conjugate to a subgroup of infinite dihedral group (also called meta-abelian group) and case 1 does not hold. And there exists an algebraic solution of degree two of the Riccati equation* (1.7)*.*

3. *Up to conjugation G is either of following finite groups: Tetrahedral group, Octahedral group or Icosahedral group, and cases 1 and 2 do not hold. And there exists an algebraic solution of degree four or six or twelve of the Riccati equation* (1.7)*.*

4. *$G = SL(2, \mathbf{C})$.*

Next, we present the eigenring of (1.6). Eigenrings are a tool, introduced by M. F. Singer, to study the decomposability of systems or operators. A general reference is [10] (and references therein). We consider two different formalisms: the matrix formalism and the operators formalism.

Consider the companion matrix of $\mathcal{P}$,

$$A_{\mathcal{P}} = -\begin{pmatrix} 0 & -1 \\ q & p \end{pmatrix} = -A, \tag{1.8}$$

and the associated matrix differential equation

$$\Phi' = A_{\mathcal{P}}\Phi = -A\Phi. \tag{1.9}$$

Let $P \in GL_2(K)$. The substitution $\Phi = P\Psi$ yields to the new matrix differential equation

$$\Psi' = -B\Psi, \qquad \text{where } B = P^{-1}P' + P^{-1}AP. \tag{1.10}$$

We say that matrices A and B are *equivalent*, denoted by $A \sim B$, if there exists a matrix $P \in GL_2(K)$ such that

$$B = P^{-1}P' + P^{-1}AP.$$

We say that differential equations (1.9) and (1.10) are *equivalent* if their coefficient matrices are aquivalent.

Assuming A and B are equivalent, i.e., $PB = P' + AP$, we can consider the case $PA = PB$. In this situation P is no longer necessarily in $GL_2(K)$. Under these assumptions we get the relation $PA = P' + AP$, which leads to the definition of eigenring. The *eigenring* of equation (1.9), denoted by $\mathcal{E}(A)$, is given by

$$\mathcal{E}(A) := \{P \in M_2(K) : P' = PA - AP\}.$$

A direct calculation shows that the product of two elements of $\mathcal{E}(A)$ also belongs to $\mathcal{E}(A)$ and $I_2 \in \mathcal{E}(A)$. Thus, $\mathcal{E}(A)$ is a $\mathbf{C}-$algebra and as a $\mathbf{C}-$vector space it has dimension ≤ 4. As a consequence, we find the following results for $\mathcal{E}(A)$:

Proposition 1.17 (Barkatou, [10])**.** *Any element $P \in \mathcal{E}(A)$ has a minimal polynomial with coefficients in the field of constants $\mathbf{C}$. In particular, when $\mathbf{C}$ is algebraically closed, each matrix $P \in \mathcal{E}(A)$ has all its eigenvalues in $\mathbf{C}$.*

Proposition 1.18 (Barkatou, [10])**.** *If two equations* (1.9) *and* (1.10) *are equivalent, their eigenrings $\mathcal{E}(A)$ and $\mathcal{E}(B)$ are isomorphic as $\mathbf{C}-$algebras. In particular, one has* $\dim_{\mathbf{C}} \mathcal{E}(A) = \dim_{\mathbf{C}} \mathcal{E}(B)$.

Regarding the operators formalism, we define the *eigenring* of the differential operator $\mathcal{P}(y) = 0$, denoted by $\mathcal{E}(\mathcal{P})$, as the set

$$\mathcal{E}(\mathcal{P}) := \{\mathcal{R} \in K[\partial] : \mathcal{R}(V(\mathcal{P})) \subseteq V(\mathcal{P})\},$$

or, equivalently,

$$\mathcal{E}(\mathcal{P}) := \{\mathcal{R}, \mathcal{S} \in K[\partial] : \mathcal{P}\mathcal{R} = \mathcal{S}\mathcal{P}\}.$$

The equivalence between $\mathcal{E}(\mathcal{P})$ and $\mathcal{E}(A)$ has been studied in [4], see also [2, 3]. In particular, we find the following result.

Proposition 1.19. *We have that $\mathcal{R} = a + b\partial_x \in \mathcal{E}(\mathcal{P})$ if and only if $P \in \mathcal{E}(A)$, being P given by*

$$P = \begin{pmatrix} a & b \\ a' - bq & a + b' - bp \end{pmatrix}.$$

Furthermore, assume G is the Galois group of equation $\mathcal{P}(y) = 0$ and $p = n(\ln w)'$, for $n \in \mathbb{Z}$ and $w \in K$, then the following statements hold:

1. *If $\dim_{\mathbf{C}} \mathcal{E}(\mathcal{P}) = 1$, then G is either irreducible or indecomposable.*
2. *If $\dim_{\mathbf{C}} \mathcal{E}(\mathcal{P}) = 2$, then either G is the additive group or G is a subgroup of the multiplicative group. Thus, we can have $y_1 \notin K$ and $y_2 \notin K$.*
3. *If $\dim_{\mathbf{C}} \mathcal{E}(\mathcal{P}) = 4$, then $G = \{I\}$. In this case we have $y_1^2 \in K$, $y_1y_2 \in K$ and $y_2^2 \in K$.*

Remark 1.20. For $P \in \mathcal{E}(A)$, we have that $P \in \mathrm{GL}(2, K)$ if and only if $\frac{a'}{a} - \frac{a}{b} + p \neq \frac{b'}{b} + \frac{b}{a}q$.

1.2.3 Symmetric powers of linear differential operators

Next, we briefly introduce some basic facts concerning the symmetric powers of differential operators. Let $\mathcal{P}(y) = 0$ be a homogeneous linear differential equation of order n. Let $\{y_1, \ldots, y_n\}$ be a fundamental system of solutions for this equation and denote by $V(\mathcal{P})$ its solution space. The differential equation $\mathfrak{sym}^m\mathcal{P}(y)$ whose solution space, denoted by $\mathrm{Sym}^m(V(\mathcal{P}))$, is spanned by the monomials of degree m in $y_1, \ldots, y_n$ is called the *m-th symmetric power* of $\mathcal{P}(y) = 0$.

To compute $\mathfrak{sym}^m\mathcal{P}(y)$, we consider $z = \prod_{i=1}^m y_i$, for $y_1, \ldots, y_n$ solutons of $\mathcal{P}(y) = 0$, where the same solution y_i can appear more than once. Taking derivatives of z and using equation $\mathcal{P}(y_i) = 0$ to simplify the expressions gives a linear differential equation for z of order at most $\binom{n+m-1}{n-1}$ (see [82]). Moreover, the Galois group G of $\mathcal{P}(y) = 0$ acts over the solutions space $\mathrm{Sym}^m(V(\mathcal{P}))$ in a natural way, which yields to another representation of G. We will study this action when $m = 2$ in Section 6.3.

In particular, we can compute the second symmetric power of the homogeneous second order linear differential equation (1.6) as follows. The second symmetric power of $\mathcal{P}$, denoted by $\mathfrak{sym}^2(\mathcal{P})$, is the third order differential equation in which $\{y_1^2, 2y_1y_2, y_2^2\}$ is a basis of solutions. We set $z = y_1y_2$, then:

$$\begin{aligned} z' &= y_1'y_2 + y_1y_2', \\ z'' &= -pz' - 2qz + 2y_1'y_2', \\ z''' &= -3pz'' - (4q + p' + 2p^2)z' - 2(q' + 2pq)z. \end{aligned}$$

So, the linear differential equation

$$\mathfrak{sym}^2(\mathcal{P})(y) := y''' + 3py'' + (4q + p' + 2p^2)y' + 2(q' + 2pq)y = 0$$

is the second symmetric power of $\mathcal{P}(y)$.

In matrix formalism, consider the differential system associated to (1.6):

$$\Phi' = A_{\mathcal{P}}\Phi, \qquad \text{for } \Phi = \begin{pmatrix} y \\ y' \end{pmatrix},$$

where $A_{\mathcal{P}}$ is the companion matrix of $\mathcal{P}$ given by (1.8). Then, the matrix differential equation associated to the differential operator $\mathfrak{sym}^2(\mathcal{P})$ is

$$\mathrm{Sym}^2(\Phi)' = \mathfrak{sym}^2(A_{\mathcal{P}}) \cdot \mathrm{Sym}^2(\Phi), \tag{1.11}$$

where

$$\mathrm{Sym}^2(\Phi) = \begin{pmatrix} y^2 \\ 2yy' \\ (y')^2 \end{pmatrix} \qquad \text{and} \qquad \mathfrak{sym}^2(A_{\mathcal{P}}) = \begin{pmatrix} 0 & 1 & 0 \\ -2q & -p & 2 \\ 0 & -q & -2p \end{pmatrix}.$$

We call this system the *second symmetric power system*.

The connection between symmetric powers of a second order linear differential equation and its integrability in the Picard–Vessiot sense comes evident in the following result (see [59, 83, 85]), which extends Liouville's Theorem 1.15.

Theorem 1.21. *Let $\mathcal{P}(y) = 0$ be a second order linear differential equation of the form* (1.6)*. Then, its associated Riccati equation* (1.7) *has an algebraic solution of degree at most m if and only if the symmetric power equation $\mathfrak{sym}^m(P)(y) = 0$ has an exponential solution.*

1.3 Darboux–Crum transformations for Schrödinger equation

In this section we survey Darboux–Crum transformations for Schrödinger equation, as stated in [25]. We also discuss about the factorization of Schrödinger operator.

Let K be a differential field with two compatible derivations ∂_x and ∂_t such that its field of constants $\mathbf{C}$ is algebraically closed and of characteristic zero. Let $u = u(x,t) \in K$ be a fixed element of K and $E \in \mathbf{C}$ a parameter.

Consider the Schrödinger operator $\mathcal{L} - E = -\partial_x^2 + u - E$ and the Schrödinger equation

$$(\mathcal{L} - E)\phi = -\phi_{xx} + (u - E)\phi = 0. \tag{1.12}$$

Let $\phi_i \neq 0$ be a solution of Schrödinger equation (1.12) for energy level E_i, for $1 \leq i \leq d$, with $E_i \neq E_j$ if $i \neq j$. Let ϕ be a general solution of (1.12). Now, we consider the potential

$$u[d] = u - 2\partial_x^2 \ln W(\phi_1, \ldots, \phi_d) \tag{1.13}$$

and the equation

$$-\phi_{xx} + u[d]\phi = E\phi. \tag{1.14}$$

Define the linear differential operator $\mathcal{D}[d]$ as:

$$\mathcal{D}[d] = \partial_x^{(d)} + s_1 \partial_x^{(d-1)} + \ldots + s_{d-1}\partial_x + s_d, \tag{1.15}$$

with $s_i = -W_i/W(\phi_1, \ldots, \phi_d)$, for $i = 1, \ldots, d$, where $W(\phi_1, \ldots, \phi_d)$ denotes de Wronskian of $\phi_1, \ldots, \phi_d$ and W_i is obtained by replacing the $(d-i)$-th column of $W(\phi_1, \ldots, \phi_d)$ by $(\partial_x^{(d)}\phi_1, \ldots, \partial_x^{(d)}\phi_d)^t$. We observe that $\mathcal{D}[d]\phi_i = 0$ for $i = 1, \ldots, d$. Then, Crum theorem states that the function

$$\phi[d] = \mathcal{D}[d]\phi \tag{1.16}$$

is a solution of (1.14). Moreover:

$$s_1 = -\partial_x(\log W(\phi_1, \ldots, \phi_d)),$$

and then $u[d] = u + 2s_{1,x} = u - 2\partial_x^2(\log W(\phi_1, \ldots, \phi_d))$.

1.3.1 Factorization of Schrödinger operator

The case of Darboux–Crum transformation for $d = 1$ is specially interesting. For this case we only consider one solution ϕ_1 of Schrödinger equation (1.12) for a particular value E_1 in order to perform the Darboux–Crum transformation. Let ϕ, as before, be a general solution of Schrödinger equation (1.12). Then, the Darboux–Crum transformation of ϕ using ϕ_1 is:

$$DT(\phi_1)\phi = \phi_x - \frac{\phi_{1,x}}{\phi_1}\phi = \phi_x - \sigma_1\phi = \widetilde{\phi}. \tag{1.17}$$

This function is a general solution of Schrödinger equation (1.12) for potential

$$DT(\phi_1)u = u - 2(\log\phi_1)_{xx} = u - 2\sigma_{1,x} = \widetilde{u}. \tag{1.18}$$

We denote the Schrödinger operator for the transformed potential by $\widetilde{\mathcal{L}} - E$ and we refer to it as the transformed Schrödinger operator. Moreover, the logarithmic derivative $\sigma_1 = (\log\phi_1)_x$ is a solution of the following Riccati equation for $E = E_1$:

$$\sigma^2 + \sigma_x = u - E. \tag{1.19}$$

Trivially from the above equation we obtain the following differential equation for σ_{xx}, which will be necessary later on in this thesis:

$$\sigma_{xx} = u_x - 2\sigma\sigma_x. \tag{1.20}$$

Notice that, after replacing expression (1.19) for u:

$$u = \sigma_1^2 + \sigma_{1,x} + E,$$

in equation (1.18) we arrive to another Riccati equation for σ_1, but this time in terms of the transformed potential:

$$\sigma^2 - \sigma_x = \widetilde{u} - E. \tag{1.21}$$

The importance of this case lies in the fact that the Riccati equations (1.19) and (1.21) enable us to factor the Schrödinger operator as a product of two conjugated differential operators of order one. In order to show this, let us consider σ belonging to a differential extension of K and the pair of conjugated opetarors $-\partial_x - \sigma$ and $\partial_x - \sigma$, then, we have

$$(-\partial_x - \sigma)(\partial_x - \sigma) = -\partial_x^2 + \sigma^2 + \sigma_x = -\partial_x^2 + u - E = \mathcal{L} - E \tag{1.22}$$

if and only if σ is a solution of the Riccati equation (1.19). In particular, we can set $\sigma = \sigma_1$.

Now, we exchange the factors in (1.22). We get:

$$(\partial_x - \sigma)(-\partial_x - \sigma) = -\partial_x^2 + \sigma^2 - \sigma_x = -\partial_x^2 + \widetilde{u} - E = \widetilde{\mathcal{L}} - E \tag{1.23}$$

if and only if σ is a solution of the Riccati equation (1.21). Again, in particular, we can set $\sigma = \sigma_1$. Hence, we also find a factorization of the transformed Schrödinger operator.

This technique of exchanging the factors of the Schrödinger operator is the so-called *transference* and was first studied by Burchnall and Chaundy in [19] to obtain pairs of commuting operators.

In this way, transference and Riccati equations (1.19) and (1.21) provide us a method to factor both Schrödinger operators $\mathcal{L} - E$ and $\widetilde{\mathcal{L}} - E$. This is relevant because the factorization of operators is very important to solve differential equations. Indeed, if we have a factorization of $\mathcal{L} = \mathcal{L}_1\mathcal{L}_2$, then, solutions of the right-hand factor $\mathcal{L}_2$ are solutions of $\mathcal{L}$ as well.

From the above, we conclude that performing a Darboux–Crum transformation of degree one to the Schrödinger operator is equivalent to apply the transference technique to that Schrödinger operator.

Chapter 2

Differential Galois Theory and Darboux transformations for integrable systems

This chapter is devoted to study how the Galois groups of the linear systems of (Zhakharov–Shabat) AKNS type behave under matrix Darboux transformations. The work exposed here is based on a joint work with Sonia Jiménez, Juan J. Morales and María Ángeles Zurro. The results that appear in this chapter are contained in the article [51].

Throughout this chapter K will denote a differential field with two compatible derivations ∂_x, ∂_t and field of constants $\mathbf{C}$ algebraically closed and of characteristic zero and $\lambda \in \mathbf{C}$ will denote a parameter.

2.1 Darboux–Crum transformations for AKNS systems

Let $u \in K$ be a fixed element of K and consider the differential ring $K_u = \mathbf{C}[u, u_x, \ldots]$ of differential polynomials in u. An *AKNS system* is a differential system of the form:

$$\begin{cases} \Phi_x & = U\Phi = (\ \lambda J + P\ )\Phi\,, \\ \Phi_t & = V\Phi = \displaystyle\sum_{j=0}^{n} V_j \lambda^j \Phi\,. \end{cases} \tag{2.1}$$

where $J = \operatorname{diag}(c_1, \ldots, c_m) \in \operatorname{Mat}_m(\mathbf{C})$ of trace zero, such that $c_i \neq c_k$ if $i \neq k$ and $\det(J) \neq 0$, $P(x,t) := P[u] \in \operatorname{Mat}_m(K_u)$ whose diagonal entries are all zero and $V_j(x,t) := V_j[u] \in \operatorname{Mat}_m(K_u)$ of trace zero, $j = 0, \ldots, n$.

If the system (2.1) satisfies the *zero curvature condition*:

$$U_t - V_x + [U, V] = 0, \tag{2.2}$$

we say that it is a *flat system*. In this case, we say that (2.1), or (U, V), is a *Lax pair* for the partial differential system equations (2.2). We denote by $(2.1)_{(U,V)}$ the equation (2.1) if it were necessary to specify the Lax pair.

It is well-known that the entries of V_j, $0 \le j \le n$, are differential polynomials in the entries of P, whose coefficients may depend on t (see, for instance, [42, p. 17]). We write $V_j[P]$ for the V_j to specify the dependence on P.

From now on in this chapter, we will assume that all the systems are flat systems.

Let $\mathcal{K}$ be a differential extension of K. A *Darboux transformation* for the system $(2.1)_{(U,V)}$ is an invertible matrix D which is a rational function of λ with coefficients in $\mathcal{K}$, such that if Φ is a solution of $(2.1)_{(U,V)}$, then $\widetilde{\Phi} := D\Phi$ is a solution of $(2.1)_{(\widetilde{U},\widetilde{V})}$, where

$$\widetilde{U} = DUD^{-1} + D_x D^{-1} \qquad \text{and} \qquad \widetilde{V} = DVD^{-1} + D_t D^{-1}. \tag{2.3}$$

The matrix D is called a *Darboux matrix*. When D is a polynomial in λ, its degree is of course the degree of that polynomial. The degree one Darboux transformations are also called *elementary Darboux transformations*.

We remark that a Darboux transformation is nothing but a special gauge transformation and it will be covariant with respect to the AKNS structure. This means that the transformed system is also an AKNS system, with the same J, P and V matrices, but depending on a new function $\widetilde{u} \in \mathcal{K}$.

Applying Liouville's Formula 1.1 to Φ and $\widetilde{\Phi} = D\Phi$ we get

$$\begin{aligned} (\det\Phi)_x &= \operatorname{tr} U \det\Phi, \qquad & (\det D \det\Phi)_x &= \operatorname{tr}\widetilde{U} \det D \det\Phi, \\ (\det\Phi)_t &= \operatorname{tr} V \det\Phi, \qquad & (\det D \det\Phi)_t &= \operatorname{tr}\widetilde{V} \det D \det\Phi, \end{aligned}$$

which leads to

$$\frac{(\det D)_x}{\det D} = \operatorname{tr}\widetilde{U} - \operatorname{tr} U \qquad \text{and} \qquad \frac{(\det D)_t}{\det D} = \operatorname{tr}\widetilde{V} - \operatorname{tr} V,$$

and, since $\operatorname{tr} U = \operatorname{tr} V = \operatorname{tr}\widetilde{U} = \operatorname{tr}\widetilde{V} = 0$, we have that $\det\Phi, \det D, \det\widetilde{\Phi} \in \mathbf{C}$.

From the above it is not difficult to prove that the set of Darboux transformations form a subgroup of the group of gauge transformations.

2.2 Elementary Darboux transformations

First, we consider the Darboux matrix of degree one, i.e., D is linear in λ. We will consider $D = D_0 + \lambda D_1$, where $D_1 = \operatorname{diag}(d_1, \ldots, d_m) \in GL_m(\mathbf{C})$. Then we have the following result, which is a slightly reformulation of Theorem 1.8 in [42]:

Proposition 2.1 (Gu, Hu and Zhou, [42])**.** *The matrix D is a Darboux matrix for the AKNS system (2.1) if and only if $\Sigma = -D_1^{-1}D_0$ satisfies*

$$\Sigma_x + [\Sigma, P + J\Sigma] = 0 \qquad \textit{and} \qquad \Sigma_t + \sum_{p=0}^{n} [\Sigma, V_p]\Sigma^p = 0\,. \tag{2.4}$$

Moreover, $\widetilde{P} = D_1(P + [J, \Sigma])D_1^{-1}$.

To solve the system (2.4) we proceed as follows:
We choose $\lambda_1, \ldots, \lambda_m$ in $\mathbf{C} \setminus \{0\}$ with $\lambda_i \neq \lambda_j$ if $i \neq j$. Take $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_m)$. For each λ_i let us take $\vec{\psi}_i$ a solution of the system (2.1) with coefficients in some differential field extension $\mathcal{K}$ of K. Assume that $\vec{\psi}_1, \ldots, \vec{\psi}_m$ are linearly independent over $\mathbf{C}$. Let us put $\Psi = (\vec{\psi}_1 \quad \ldots \quad \vec{\psi}_m)$. Then the matrix $\Sigma = \Psi\Lambda\Psi^{-1} = (\sigma_{kl})$ satisfies the system (2.4) and $\lambda I - \Sigma$ is an elementary Darboux transformation (see Theorem 1.3 in [80], and also [11]).

The family given by $\lambda \to D = D(\lambda) = \lambda I - \Sigma$, for Σ constructed as before, has the evolution given by (2.4), since $D_x = -\Sigma_x$ and $D_t = -\Sigma_t$.

Next, we proceed to compute the inverse of D. For that, observe that when Λ is of trace zero the characteristic polynomial of Λ, say $p_\Lambda(X) = \det(XI - \Lambda) = a_0 + a_1X + \cdots + a_{m-2}X^{m-2} + X^m \in \mathbf{C}[X]$, has the property $p_\Lambda(\lambda) = \det(D)$. Hence the characteristic polynomial of D is

$$p_D(X) = (-1)^m p_\Lambda(\lambda - X).$$

So, $\det D|_{\lambda=\lambda_i} = 0$, but $p_D(\lambda_i) \neq 0$. Furthermore, we have:

$$p_D(X) = \lambda^m X^m - m\lambda^{m-1}X^{m-1} + \sum_{p=0}^{m-2} c_p(\lambda)X^p\,, \quad \deg_\lambda c_p(\lambda) \leq m - p - 2.$$

Since $p_D(D) = 0$, we get the formula for the inverse of D:

$$c_0(\lambda)D^{-1} = -\lambda^m D^{m-1} + m\lambda^{m-1}D^{m-2} - \sum_{p=1}^{m-2} c_p(\lambda)D^{p-1}.$$

2.2.1 Galois groups and elementary Darboux transformations

Let D be a Darboux transformation of degree one for the AKNS system (2.1). Let Σ be a solution of (2.4) constructed for some diagonal constant matrix Λ as before. Hence we get the new AKNS system:

$$\begin{cases} \widetilde{\Phi}_x &= \widetilde{U}\widetilde{\Phi} = (\ \lambda J + \widetilde{P}\ )\widetilde{\Phi}\,, \\ \widetilde{\Phi}_t &= \widetilde{V}\widetilde{\Phi} = \displaystyle\sum_{j=0}^{n} \widetilde{V}_j \lambda^j \widetilde{\Phi}\,. \end{cases} \tag{2.5}$$

Now we choose $\lambda \in \mathbf{C}$ such that $p_\Lambda(\lambda) \neq 0$. Let us take $\Sigma = (\sigma_{kl})$ a solution of (2.4). Then the field $\widetilde{K} = K(\sigma_{kl}) = K(\Sigma)$ is the field of coefficients of the system

(2.5). Let $\Phi = (\phi_{kl})$ be a fundamental solution of (2.1) and let $L = K(\phi_{kl}) = K(\Phi)$ be a Picard–Vessiot field for (2.1). Let $\widetilde{\Phi} = (\widetilde{\phi}_{kl})$ be a fundamental solution of (2.5) and let $\widetilde{L} = \widetilde{K}(\widetilde{\phi}_{kl}) = \widetilde{K}(\widetilde{\Phi})$ be a Picard–Vessiot field for (2.5). Then we have the field commutative diagram:

$$\begin{tikzcd}
& \widetilde{L} & \\
L \arrow[ur] & & \widetilde{K} \arrow[ul, "\widetilde{G}"'] \\
& L\cap\widetilde{K} \arrow[ul, "H"'] \arrow[ur] & \\
& K \arrow[uul, "G"] \arrow[u] &
\end{tikzcd} \tag{2.6}$$

Let G be the Galois group of the system (2.1), and consider the Galois group of the differential field extension $L \cap \widetilde{K} \subset L$, say H. Let be $\widetilde{G}$ the Galois group of (2.5). Then we have the following statement:

Theorem 2.2. *The Galois group $\widetilde{G}$ is isomorphic to the subgroup $H = Gal(L/L\cap\widetilde{K})$ of the Galois group G.*

In order to prove this statement, we need the following lemma. For the convenience of the reader, we reproduce it.

Lemma 2.3 (Lemma 5.10 in [54])**.** *Let L be a Picard–Vessiot extension of K (characteristic 0 and algebraically closed constant field). Let $L_1 = L\langle z\rangle$ be an extension of L with no new constants. Write $K_1 = K\langle z\rangle$. Then L_1 is a Picard–Vessiot extension of K_1, and its differential Galois group, G_1, is isomorphic to an algebraic subgroup of the differential Galois group of L over K, G, namely the subgroup leaving $L \cap K_1$ fixed.*

Proof of 2.2. First, we note that $\widetilde{K}(D) = \widetilde{K}(\Sigma) = \widetilde{K}$. Hence, we have

$$\widetilde{L} = \widetilde{K}(\widetilde{\Phi}) = \widetilde{K}(D\Phi) = \widetilde{K}(\Phi) = K(\Sigma, \Phi) = K(\Phi)(\Sigma) = L(\Sigma) = L(\sigma_{kl}).$$

As the derivatives of the entries of matrix Σ are defined rationally by means of equation (2.4), we have that $\partial_x(\sigma_{kl}) \in L(\sigma_{kl})$ and $\partial_t(\sigma_{kl}) \in L(\sigma_{kl})$, thus $L(\sigma_{kl}) = L\left\langle\sigma_{kl}\right\rangle$. Then, we apply Lemma 2.3 and using induction over σ_{kl} we obtain that (see diagram (2.6))

$$\widetilde{G} \simeq H \simeq Gal(L/L\cap\widetilde{K}) \subset G$$

and this conclude the proof. $\square$

This theorem means that the transformed system is at least as integrable (in the Picard–Vessiot sense) as the initial system was. In particular, if the initial system is integrable, the transformed system is integrable as well.

This theorem has two immediate consequences:

Corollary 2.4. *We have the equivalence:* $G \simeq \widetilde{G}$ *if and only if* $K = L \cap \widetilde{K}$.

As in Chapter 1, let G^0 denotes the connected identity component of G.

Corollary 2.5. *If* $K \subset \widetilde{K}$ *is an algebraic extension, then* $G^0 \simeq \widetilde{G}^0$.

Proof. If $K \subset \widetilde{K}$ is an algebraic extension, $K \subset L \cap \widetilde{K}$ is also algebraic. Then, $G_0 \subseteq H$ and, since $H \simeq \widetilde{G}$, we conclude that $G^0 \simeq \widetilde{G}^0$. □

Remark 2.6. A geometric interpretation of Theorem 2.2 can be done by means of torsors (see Subsection 1.1.2). Let R and $\widetilde{R}$ be Picard–Vessiot rings of (2.1) and (2.5) respectively. Consider S the multiplicative closed system of $R[z]$ given by the powers of $p_\Lambda(z) = \det(zI - \lambda)$, and also $S_\lambda = \{p_\Lambda(\lambda)^n : \; n \in \mathbb{N}\}$. Then

$$(S^{-1}R[z])/S^{-1}(z-\lambda) \simeq S^{-1}(R[z]/(z-\lambda)) \simeq S_\lambda^{-1}R,$$

since $(z - \lambda) \cap S = \emptyset$ because $p_\Lambda(\lambda) \neq 0$. Now we have:

$$S_\lambda^{-1}R \otimes_K \widetilde{K} \simeq S_\lambda^{-1}\widetilde{R},$$

because $S_\lambda^{-1}\widetilde{K} = \widetilde{K}$, since it is a field. But $\widetilde{R}$ is a domain, then $S_\lambda^{-1}\widetilde{R} \simeq \widetilde{R}$, because they are simple rings. So, at the end we get the following formula:

$$S_\lambda^{-1}R \otimes_K \widetilde{K} \simeq \widetilde{R}.$$

Hence $Z_{\widetilde{L}} \simeq Z_L \otimes_K \widetilde{K}$, where Z and $\widetilde{Z}$ are the maximal spectra of R and $\widetilde{R}$ respectively.

2.3 Darboux matrices of higher degree

For the construction of the higher degree Darboux matrices we follow [42]. Then we will obtain some galoisian consequences of these constructions.

The composition of d Darboux transformations of degree one is a Darboux transformation of degree d. Although, we can construct Darboux transformations of degree d directly. To do this, let us consider a $m \times m$ matrix of the form:

$$D(x,t,\lambda) = \sum_{j=0}^{d} \widetilde{D}_j(x,t)\lambda^j = \widetilde{D}_d \left(\sum_{j=0}^{d-1} D_j(x,t)\lambda^j + \lambda^d I \right), \tag{2.7}$$

where $\widetilde{D}_d = \operatorname{diag}(d_1, \ldots, d_m) \in GL_m(\mathbf{C})$. As seen before, a Darboux matrix of degree one of the form $D(x,t,\lambda) = D_1(\lambda I - \Sigma)$ can be constructed. Suppose $\Sigma = \Psi\Lambda\Psi^{-1}$, then $\Sigma\Psi = \Psi\Lambda$ is equivalent to $D(x,t,\lambda_i)\psi_i = 0$. This can be generalized to Darboux matrices of degree d as follows.

For $i = 1, \ldots, md$, let us take a solution $\vec{\psi}_i$ of the system (2.1) for $\lambda = \lambda_i \in \mathbf{C} \setminus \{0\}$. As before, assume that $\vec{\psi}_1, \ldots, \vec{\psi}_{md}$ are linearly independent over $\mathbf{C}$. Let us put $\Psi = (\vec{\psi}_1 \quad \ldots \quad \vec{\psi}_{md})$.

In order to simplify the notation, hereafter we write $\Psi = (\psi_1 \quad \dots \quad \psi_{md})$. Consider the $md \times md$ matrix:

$$\Gamma_d = \begin{pmatrix} \psi_1 & \psi_2 & \dots & \psi_{md} \\ \lambda_1\psi_1 & \lambda_2\psi_2 & \dots & \lambda_{md}\psi_{md} \\ \vdots & \vdots & \ddots & \vdots \\ \lambda_1^{d-1}\psi_1 & \lambda_2^{d-1}\psi_2 & \dots & \lambda_{md}^{d-1}\psi_{md} \end{pmatrix}. \tag{2.8}$$

The system $D(x,t,\lambda_i)\psi_i = 0$, $i = 1, \dots, md$, is equivalent to:

$$\sum_{j=0}^{d-1} D_j(x,t)\lambda_i^j \psi_i = -\lambda_i^d \psi_i\,, \tag{2.9}$$

for $i = 1, \dots, md$, which can be written as:

$$(D_0, D_1, \dots, D_{d-1})\Gamma_d = -(\lambda_1^d\psi_1, \lambda_2^d\psi_2, \dots, \lambda_{md}^d\psi_{md}). \tag{2.10}$$

When $\det\Gamma_d \neq 0$, this system has a unique solution $(D_0, D_1, \dots, D_{d-1})$, i.e., a unique $m \times m$ matrix $D(x,t,\lambda)$ in the form (2.7) such that $D(x,t,\lambda_i)\psi_i = 0$, for $i = 1, \dots, md$.

We denote this matrix by $D(\psi_1, \dots, \psi_{md}, \lambda)$ to indicate that it is constructed from $\psi_1, \dots, \psi_{md}$.

The following result states that this matrix is a Darboux matrix and that it can be decomposed, under some assumptions, as a product of two Darboux matrices of lower degree (see [42], Theorem 1.12). Let us denote:

$$\begin{cases} \widetilde{\Phi}_x & = \widetilde{U}\widetilde{\Phi} = (\ \lambda J + \widetilde{P}\ )\widetilde{\Phi}\,, \\ \widetilde{\Phi}_t & = \widetilde{V}\widetilde{\Phi} = \displaystyle\sum_{j=0}^{n} \widetilde{V}_j \lambda^j \widetilde{\Phi}\,. \end{cases} \tag{2.11}$$

for the transformed system.

Proposition 2.7. *Given $\lambda_1, \dots, \lambda_{md} \in \mathbf{C}$. Let ψ_i be a column solution of (2.1), $1 \leq i \leq md$, and Γ_d be defined by (2.8). Suppose that $\det\Gamma_d \neq 0$, then the following statements hold:*

1. *There exist a unique matrix $D(\psi_1, \dots, \psi_{md}, \lambda)$ in the form (2.7) such that*

$$D(\psi_1, \dots, \psi_{md}, \lambda_i)\psi_i = 0, \qquad 1 \leq i \leq md. \tag{2.12}$$

   We say that $D(\psi_1, \dots, \psi_{md}, \lambda)$ is a Darboux matrix of degree d for (2.1).

2. *If $\det\Gamma_{d-1} \neq 0$, then this Darboux matrix of degree d can be decomposed as:*

$$D(\psi_1, \dots, \psi_{md}, \lambda) = \tag{2.13}$$

$$D(D(\psi_1, \dots, \psi_{m(d-1)}, \lambda_{m(d-1)+1})\psi_{m(d-1)+1}, \dots, D(\psi_1, \dots, \psi_{m(d-1)}, \lambda_{md})\psi_{md}, \lambda)\cdot$$
$$\cdot D(\psi_1, \dots, \psi_{m(d-1)}, \lambda).$$

   Where the first matrix on the right hand side of this equality is a Darboux matrix of degree one and the second one is a Darboux matrix of degree $d-1$.

3. *If* $\det \Gamma_i \neq 0$ *for* $i = 1, \ldots, d-1$, *the Darboux matrix* $D(\psi_1, \ldots, \psi_{md}, \lambda)$ *of degree* d *can be decomposed as a product of* d *Darboux matrices of degree one.*

4. *Let us take* D *a decomposable Darboux transformation, that is* $D = \widetilde{D}_d(\lambda I - \Sigma_d) \cdots (\lambda I - \Sigma_1)$. *Then the matrix* $\widetilde{P} = \widetilde{D}_d(P - [J, D_{d-1}])\widetilde{D}_d^{-1}$ *satisfies* $\widetilde{U} = \lambda J + \widetilde{P}$ *in (2.11).*

The proof of this proposition is an easy extension of the argument in [42].

From the above proposition and Theorem 2.2, we arrive to the following:

Theorem 2.8. *Let* G *be the Galois group of system (2.1). Let* $D = D_d \cdots D_1$ *be a decomposable Darboux transformation of degree* d. *If* G_i *is the Galois group of system (2.1) transformed by* $\widetilde{D_i} = D_i \cdots D_1$, *then we have the chain of groups:*

$$G_d \subset G_{d-1} \subset \cdots \subset G_1 \subset G.$$

Remark 2.9. We observe that G_d will be the Galois group of the transformed system under D for $\lambda \in \mathbf{C} \setminus \{\det(D(\lambda)) = 0\}$.

Now we will consider the case $d = 2$.

Suppose $D(\lambda) = D(\psi_1, \ldots, \psi_{2m}, \lambda)$ is a Darboux matrix of degree two with $\widetilde{D}_2 = I$ and decomposable as a product of two Darboux matrices of degree one of the form (2.13):

$$D(\lambda) = D(\widetilde{\psi}_{m+1}, \ldots, \widetilde{\psi}_{2m}, \lambda) \cdot D(\psi_1, \ldots, \psi_m, \lambda),$$

where $\widetilde{\psi}_i = D(\psi_1, \ldots, \psi_m, \lambda_i)\psi_i$, for $i = m+1, \ldots, 2m$. The matrix $D^{(1)} = D(\psi_1, \ldots, \psi_m, \lambda)$ has the form:

$$D^{(1)} = \lambda I - \Sigma_1 = \lambda I - \Psi^{(1)}\Lambda_1(\Psi^{(1)})^{-1},$$

where $\Psi^{(1)} = (\psi_1, \ldots, \psi_m)$ for some $\Lambda_1 = \mathrm{diag}(\lambda_1, \ldots, \lambda_m)$ (see (2.4) and the construction of a Darboux transformation for $d = 1$).

For $i = m+1, \ldots, 2m$, we denote by $\widetilde{\psi}_i$ the transformed of ψ_i, that is:

$$\widetilde{\psi}_i = D(\psi_1, \ldots, \psi_m, \lambda_i)\psi_i = (\lambda_i I - \Sigma_1)\psi_i\ .$$

So,

$$\widetilde{\Psi}^{(2)} = (\widetilde{\psi}_{m+1}, \ldots, \widetilde{\psi}_{2m}) = \Psi^{(2)}\Lambda_2 - \Sigma_1\Psi^{(2)} = \Sigma_2\Psi^{(2)} - \Sigma_1\Psi^{(2)} = (\Sigma_2 - \Sigma_1)\Psi^{(2)},$$

where $\Psi^{(2)} = (\psi_{m+1}, \ldots, \psi_{2m})$ and $\Lambda_2 = \mathrm{diag}(\lambda_{m+1}, \ldots, \lambda_{2m})$ such that $p_{\Lambda_1}(\lambda_i) \neq 0$, for the new considered λ_i. Hence:

$$\widetilde{D}^{(2)} = D(\widetilde{\psi}_{m+1}, \ldots, \widetilde{\psi}_{2m}, \lambda) = \lambda I - \widetilde{\Sigma}_2 = \lambda I - \widetilde{\Psi}^{(2)}\Lambda_2(\widetilde{\Psi}^{(2)})^{-1}.$$

Therefore,

$$D(\lambda) = \widetilde{D}^{(2)}D^{(1)}.$$

The previous procedure can be done exchanging the matrices Λ_1 and Λ_2, whenever their characteristic polynomials are coprime. More precisely, we consider the Darboux matrix $D'(\lambda) = D(\psi_{m+1}, \ldots, \psi_{2m}, \psi_1, \ldots, \psi_m, \lambda)$ of degree two with $\widetilde{D}'_2 = I$. This matrix is also decomposable as a product of two Darboux matrices of degree one:

$$D'(\lambda) = D(\widetilde{\psi}_1, \ldots, \widetilde{\psi}_m, \lambda) \cdot D(\psi_{m+1}, \ldots, \psi_{2m}, \lambda),$$

where $\widetilde{\psi}_i = D(\psi_{m+1}, \ldots, \psi_{2m}, \lambda_i)\psi_i$ for $i = 1, \ldots, m$. In this case, we have that:

$$\begin{aligned} D^{(2)} &= D(\psi_{m+1}, \ldots, \psi_{2m}, \lambda) = \lambda I - \Sigma_2 = \lambda I - \Psi^{(2)}\Lambda_2(\Psi^{(2)})^{-1}, \\ \widetilde{D}^{(1)} &= D(\widetilde{\psi}_1, \ldots, \widetilde{\psi}_m, \lambda) = \lambda I - \widetilde{\Sigma}_1 = \lambda I - \widetilde{\Psi}^{(1)}\Lambda_2(\widetilde{\Psi}^{(1)})^{-1}, \end{aligned}$$

where $\widetilde{\Psi}^{(1)} = (\widetilde{\psi}_1, \ldots, \widetilde{\psi}_m) = \Psi^{(1)}\Lambda_1 - \Sigma_2\Psi^{(1)} = \Sigma_1\Psi^{(1)} - \Sigma_2\Psi^{(1)} = (\Sigma_1 - \Sigma_2)\Psi^{(1)}$. Therefore,

$$D'(\lambda) = \widetilde{D}^{(1)}D^{(2)}.$$

Both matrices $D(\lambda)$ and $D'(\lambda)$ satisfy:

$$D(\lambda_i)\psi_i = 0 \qquad \text{and} \qquad D'(\lambda_i)\psi_i = 0,$$

for $i = 1, \ldots, 2m$. So, by part (1) of Proposition 2.7, we have that $D(\lambda) = D'(\lambda)$ and

$$\begin{aligned} \widetilde{D}^{(2)}D^{(1)} &= \widetilde{D}^{(1)}D^{(2)}, \\ (\lambda I - \widetilde{\Sigma}_2)(\lambda I - \Sigma_1) &= (\lambda I - \widetilde{\Sigma}_1)(\lambda I - \Sigma_2), \\ \lambda^2 I - \lambda(\Sigma_1 + \widetilde{\Sigma}_2) + \widetilde{\Sigma}_2\Sigma_1 &= \lambda^2 I - \lambda(\Sigma_2 + \widetilde{\Sigma}_1) + \widetilde{\Sigma}_1\Sigma_2. \end{aligned}$$

Since $\lambda = 0 \in \mathbf{C} \setminus \{\det(D(\lambda)) = 0\}$, we have $\widetilde{\Sigma}_2\Sigma_1 = \widetilde{\Sigma}_1\Sigma_2$, and then $\Sigma_1 + \widetilde{\Sigma}_2 = \widetilde{\Sigma}_1 + \Sigma_2$. Then we have the exchange formula:

$$\widetilde{\Sigma}_2\Sigma_1 = (\Sigma_1 - \Sigma_2)\Sigma_2(\Sigma_1 - \Sigma_2)^{-1}\Sigma_1 = (\Sigma_1 - \Sigma_2)\Sigma_1(\Sigma_1 - \Sigma_2)^{-1}\Sigma_2 = \widetilde{\Sigma}_1\Sigma_2. \quad (2.14)$$

That gives us the following classical result (see, for example, [42]):

Corollary 2.10 (Theorem of permutability)**.** *Let $\Lambda_1 = \operatorname{diag}(\lambda_1, \ldots, \lambda_m)$ and $\Lambda_2 = \operatorname{diag}(\lambda_{m+1}, \ldots, \lambda_{2m})$ be two invertible matrices, with $\lambda_i \neq \lambda_j$ if $i \neq j$. Let $\Phi[\lambda_i] = (\psi_{i1} \ \ldots \ \psi_{im})$ be a solution of (2.1) for $\lambda = \lambda_i$, $1 \leq i \leq 2m$. For $i_0 \in \{1, \ldots, m\}$ consider $\Psi^{(1)}_{i_0} = (\psi_{1i_0} \ \ldots \ \psi_{mi_0})$ and $\Psi^{(2)}_{i_0} = (\psi_{m+1,i_0} \ \ldots \ \psi_{2m,i_0})$. Let us put $D^{(j)} = \lambda I - \Psi^{(j)}_{i_0}\Lambda_j(\Psi^{(j)}_{i_0})^{-1} = \lambda I - \Sigma_j$ and the corresponding transformed system $(\tilde{\mathfrak{s}})^{(j)}$, $j = 1, 2$.*

Let $\widetilde{\Phi}^{(j)}[\lambda_i] = (\widetilde{\psi}_{i1} \ \ldots \ \widetilde{\psi}_{im})$ be a solution of $(\tilde{\mathfrak{s}})^{(j)}$ for $\lambda = \lambda_i$, $1 \leq i \leq 2m$. For $i_0 \in \{1, \ldots, m\}$ consider $\widetilde{\Psi}^{(2)}_{i_0} = (\widetilde{\psi}_{1i_0} \ \ldots \ \widetilde{\psi}_{mi_0})$ and $\widetilde{\Psi}^{(1)}_{i_0} = (\widetilde{\psi}_{m+1,i_0} \ \ldots \ \widetilde{\psi}_{2m,i_0})$. Let us put $\widetilde{D}^{(j)} = \lambda I - \widetilde{\Psi}^{(j)}_{i_0}\Lambda_j(\widetilde{\Psi}^{(j)}_{i_0})^{-1} = \lambda I - \widetilde{\Sigma}_j$, $j = 1, 2$. Suppose that:

$$\det \begin{pmatrix} \Psi^{(1)}_{i_0} & \Psi^{(2)}_{i_0} \\ \Psi^{(1)}_{i_0}\Lambda_1 & \Psi^{(2)}_{i_0}\Lambda_2 \end{pmatrix} \neq 0.$$

Then, for every fundamental solution Φ *of (2.1) we have:*

$$\widetilde{D}^{(2)}D^{(1)}\Phi = \widetilde{D}^{(1)}D^{(2)}\Phi \qquad \textit{and} \qquad P^{(1,2)} = P^{(2,1)},$$

where $P^{(1,2)} = P - [J, -\Sigma_1 - \widetilde{\Sigma}_2]$ *and* $P^{(2,1)} = P - [J, -\Sigma_2 - \widetilde{\Sigma}_1]$.

The theorem of permutability can be expressed by the following diagram:

$$\begin{tikzcd}
& (P^{(1)},\Phi^{(1)}) \arrow[dr, "\widetilde{D}^{(2)}", "\Lambda_2"'] & \\
(P,\Phi) \arrow[ur, "D^{(1)}", "\Lambda_1"'] \arrow[dr, "\Lambda_2", "D^{(2)}"'] & & (P^{(1,2)},\Phi^{(1,2)}) = (P^{(2,1)},\Phi^{(2,1)}) \\
& (P^{(2)},\Phi^{(2)}) \arrow[ur, "\Lambda_1", "\widetilde{D}^{(1)}"'] &
\end{tikzcd} \tag{2.15}$$

Now, if G is the Galois group of (2.1), and $\widetilde{G}_1$ (resp. $\widetilde{G}_2$) is the Galois group of (2.5) for $D = \widetilde{D}^{(1)}$ (resp. $D = \widetilde{D}^{(2)}$), we can compare the exchanging effect of diagram (2.15) on Galois groups. In fact we have,

Corollary 2.11. *The Galois group* G *of the system (2.1) after the Darboux transform* $\widetilde{D}^{(2)}D^{(1)}$*, say* $\widetilde{G}_{12}$*, is isomorphic to Galois group of the system (2.1) after the Darboux transform* $\widetilde{D}^{(1)}D^{(2)}$*, say* $\widetilde{G}_{21}$*, both of them as subgroups of the Galois group* G *of (2.1).*

Moreover we find the corresponding commutative diagram to (2.15) for Galois groups:

$$\begin{tikzcd}
& \widetilde{G}_2 \arrow[dl, "D^{(1)}"'] & \\
G & & \widetilde{G}_{12} = \widetilde{G}_{21} \arrow[ul, "\widetilde{D}^{(2)}"'] \arrow[dl, "\widetilde{D}^{(1)}"] \\
& \widetilde{G}_1 \arrow[ul, "D^{(2)}"] &
\end{tikzcd}$$

2.4 The KdV equation

We apply the above general results to the classical solitonic solutions of the Korteweg de Vries equation (KdV).

Let E be a complex parameter. Consider the system:

$$\Phi^0_x = U^0\Phi^0 = \begin{pmatrix} 0 & 1 \\ u-E & 0 \end{pmatrix}\Phi^0, \qquad \Phi^0_t = V^0\Phi^0 = \begin{pmatrix} A & B \\ C & -A \end{pmatrix}\Phi^0, \tag{2.16}$$

where $A = A[u]$, $B = B[u]$ and $C = C[u]$ are differential polynomials of u and the parameter E.

The zero curvature condition of the system is

$$U_t^0 - V_x^0 + [U^0, V^0] = 0,$$

hence

$$A = -\frac{1}{2}B_x, \qquad C = (u - E)B - \frac{1}{2}B_{xx}, \tag{2.17}$$

and so we have:

$$u_t = 2(u - E)B_x + u_x B - \frac{1}{2}B_{xxx}. \tag{2.18}$$

In particular, if

$$A = -u_x, \qquad B = 2u + 4E, \qquad C = 2u^2 + 2uE - 4E^2 - u_{xx}, \tag{2.19}$$

the zero curvature condition (2.18) becomes the KdV equation:

$$u_t - 6uu_x + u_{xxx} = 0. \tag{2.20}$$

Thus, the linear system associated to the KdV equation is:

$$\begin{cases} \Phi_x^0 &= U^0\Phi^0 = \begin{pmatrix} 0 & 1 \\ u - E & 0 \end{pmatrix} \Phi^0\,, \\ \Phi_t^0 &= V^0\Phi^0 = \begin{pmatrix} -u_x & 2u + 4E \\ 2u^2 + 2uE - 4E^2 - u_{xx} & u_x \end{pmatrix} \Phi^0\,. \end{cases} \tag{2.21}$$

Take $\Phi^0 = (\phi_1 \;\; \phi_2)^t$ a column solution of (2.16), then it satisfies the Schrödinger equation with spectral parameter E:

$$(-\partial_x^2 + u)\phi_1 = E\phi_1. \tag{2.22}$$

Now, for $\lambda^2 + E = 0$, the system (2.16) is transformed into

$$\begin{cases} \Phi_x &= U\Phi, \\ \Phi_t &= V\Phi\,, \end{cases} \tag{2.23}$$

where

$$R = \begin{pmatrix} \lambda & 1 \\ -1 & 0 \end{pmatrix},$$

$$U = RU^0R^{-1} = \begin{pmatrix} \lambda & -u \\ -1 & -\lambda \end{pmatrix} = \lambda \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} + \begin{pmatrix} 0 & -u \\ -1 & 0 \end{pmatrix},$$

$$V = RV^0R^{-1} = \begin{pmatrix} \lambda B - A & \lambda^2 B - 2\lambda A - C \\ -B & -\lambda B + A \end{pmatrix},$$

and $\Phi = R\Phi^0$. This system is an AKNS system. Moreover its zero curvature condition are equations (2.17) and (2.18) with $E = -\lambda^2$.

Then, for

$$A = -u_x, \qquad B = 2u - 4\lambda^2, \qquad C = 2u^2 - 2u\lambda^2 - 4\lambda^4 - u_{xx},$$

we have the AKNS system:

$$\begin{cases} \Phi_x &= U\Phi = \begin{pmatrix} \lambda & -u \\ -1 & -\lambda \end{pmatrix} \Phi, \\ \Phi_t &= V\Phi = \begin{pmatrix} 2\lambda u - 4\lambda^3 + u_x & 4\lambda^2 u + u_{xx} - 2u^2 + 2\lambda u_x \\ 4\lambda^2 - 2u & -2\lambda u + 4\lambda^3 - u_x \end{pmatrix} \Phi \,, \end{cases} \tag{2.24}$$

whose zero curvature condition is the KdV equation (2.20).

Let ϕ_0 be a solution of (2.22) for $E = E_0$, then, for $E_0 + \lambda_0^2 = 0$ we have:

$$\Phi_0 = R_0 \Phi_0^0 = \begin{pmatrix} \lambda_0\phi_0 + \phi_{0,x} \\ -\phi_0 \end{pmatrix}, \tag{2.25}$$

with $\Phi_0^0 = \begin{pmatrix} \phi_0 \\ \phi_{0,x} \end{pmatrix}$ a column solution of (2.16) and $R_0 = \begin{pmatrix} \lambda_0 & 1 \\ -1 & 0 \end{pmatrix}$. Hence, from a solution of (2.22) we can obtain a solution of (2.23) whenever $E_0 + \lambda_0^2 = 0$. And it is easy to check the following remark:

Remark 2.12. Let $\begin{pmatrix} \alpha \\ \beta \end{pmatrix}$ be a column solution of (2.23) for $\lambda = \lambda_0$ and $\beta \neq 0$. Then $\begin{pmatrix} \alpha + 2\lambda_0\beta \\ \beta \end{pmatrix}$ is a column solution of (2.23) for $\lambda = -\lambda_0$.

Let put for $\lambda_0 \neq 0$:

$$\Lambda = \begin{pmatrix} \lambda_0 & 0 \\ 0 & -\lambda_0 \end{pmatrix} \qquad \text{and} \qquad \Psi = \begin{pmatrix} \lambda_0\phi_0 + \phi_{0,x} & \phi_{0,x} - \lambda_0\phi_0 \\ -\phi_0 & -\phi_0 \end{pmatrix}$$

for ϕ_0 a non trivial solution of the Schrödinger equation (2.22) for $E_0 = -\lambda_0^2 \neq 0$. A Darboux matrix $D = J(\lambda I - \Sigma)$, with $\Sigma = \Psi\Lambda\Psi^{-1}$, for the system (2.23) is:

$$D = J(\lambda I - \Sigma) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \cdot \begin{pmatrix} \lambda - \sigma_0 & -\sigma_0^2 + \lambda_0^2 \\ 1 & \lambda + \sigma_0 \end{pmatrix} = \begin{pmatrix} \lambda - \sigma_0 & -\sigma_0^2 + \lambda_0^2 \\ -1 & -\lambda - \sigma_0 \end{pmatrix}, \tag{2.26}$$

where $\sigma_0 = \frac{\phi_{0,x}}{\phi_0}$ (see [42], section 1.4.1 for more details on the construction of Darboux transformations for KdV equation). We notice that σ_0 is a solution of the Riccati equations for $\lambda = \lambda_0$:

$$\begin{aligned} \sigma_x &= (u - E) - \sigma^2 = (u + \lambda^2) - \sigma^2, \\ \sigma_t &= C - 2A\sigma - B\sigma^2 = (u + \lambda^2 - \sigma^2)B - \frac{1}{2}B_{xx} + \sigma B_x = \sigma_x B + \sigma B_x - \frac{1}{2}B_{xx} \\ &= 2u\sigma_x - 4\lambda^2\sigma_x + 2\sigma u_x - u_{xx}. \end{aligned}$$

Then for $\widetilde{\Phi} = D\Phi$:

$$\widetilde{\Phi} = D\Phi = \begin{pmatrix} (\lambda^2 - \lambda\sigma_0 + \sigma_0^2 - \lambda_0^2)\phi + (\lambda - \sigma_0)\phi_x \\ \sigma_0\phi - \phi_x \end{pmatrix}, \tag{2.27}$$

we have the following AKNS system:

$$\begin{cases} \widetilde{\Phi}_x = \widetilde{U}\widetilde{\Phi} & = (DUD^{-1} + D_x D^{-1})\widetilde{\Phi} \\ & = \begin{pmatrix} \lambda & u - 2(\sigma_0^2 - \lambda_0^2) \\ -1 & -\lambda \end{pmatrix} \widetilde{\Phi} = \begin{pmatrix} \lambda & -\widetilde{u} \\ -1 & -\lambda \end{pmatrix} \widetilde{\Phi}, \\ \widetilde{\Phi}_t = \widetilde{V}\widetilde{\Phi} & = (DVD^{-1} + D_t D^{-1})\widetilde{\Phi} \\ & = \begin{pmatrix} \lambda B[\widetilde{u}] - A[\widetilde{u}] & \lambda^2 B[\widetilde{u}] - 2\lambda A[\widetilde{u}] - C[\widetilde{u}] \\ -B[\widetilde{u}] & -\lambda B[\widetilde{u}] + A[\widetilde{u}] \end{pmatrix} \widetilde{\Phi}, \end{cases} \tag{2.28}$$

where $\widetilde{u} = -u + 2(\sigma_0^2 - \lambda_0^2) = -u + 2(u - \sigma_{0,x}) = u - 2\sigma_{0,x}$ (i.e., the classical Darboux–Crum transformation as shown in (1.18)) and $A[\widetilde{u}]$, $B[\widetilde{u}]$, $C[\widetilde{u}]$ satisfy the relations (2.17) and (2.18).

Now take the trivial potential for (2.22), $u = 0$. Then, for $E_0 = -\lambda_0^2 \neq 0$, the solution of (2.22)

$$\phi_0 = \cosh(\lambda_0 x - 4\lambda_0^3 t) \tag{2.29}$$

gives the solutions of (2.24):

$$\psi_{0,1} = \begin{pmatrix} \lambda_0\phi_0 + \phi_{0,x} \\ -\phi_0 \end{pmatrix} \qquad \text{and} \qquad \psi_{0,2} = \begin{pmatrix} -\lambda_0\phi_0 + \phi_{0,x} \\ -\phi_0 \end{pmatrix}, \tag{2.30}$$

for $\lambda = \lambda_0$ and $\lambda = -\lambda_0$ respectively. So, applying the Darboux transform (2.26) with

$$\Lambda_0 = \begin{pmatrix} \lambda_0 & 0 \\ 0 & -\lambda_0 \end{pmatrix} \qquad \text{and} \qquad \Psi_0 = \begin{pmatrix} \psi_{0,1} & \psi_{0,2} \end{pmatrix}$$

the system (2.24) becomes:

$$\begin{cases} \widetilde{\Phi}_x & = \widetilde{U}\widetilde{\Phi} = \begin{pmatrix} \lambda & -2(\sigma_0^2 - \lambda_0^2) \\ -1 & -\lambda \end{pmatrix} \widetilde{\Phi} = \begin{pmatrix} \lambda & -\widetilde{u} \\ -1 & -\lambda \end{pmatrix} \widetilde{\Phi}, \\ \widetilde{\Phi}_t & = \widetilde{V}\widetilde{\Phi} = \begin{pmatrix} 2\lambda\widetilde{u} - 4\lambda^3 + \widetilde{u}_x & 4\lambda^2\widetilde{u} + \widetilde{u}_{xx} - 2\widetilde{u}^2 + 2\lambda\widetilde{u}_x \\ 4\lambda^2 - 2\widetilde{u} & -2\lambda\widetilde{u} + 4\lambda^3 - \widetilde{u}_x \end{pmatrix} \widetilde{\Phi}, \end{cases} \tag{2.31}$$

where $\widetilde{u} = 2(\sigma_0^2 - \lambda_0^2) = -2\lambda_0^2 \operatorname{sech}^2(\lambda_0 x - 4\lambda_0^3 t)$ is the one soliton solution of the KdV equation.

Now, we consider the field of coefficients $K = \mathbf{C}(x,t)$ and the solution (2.29) of (2.22), then

$$\sigma_0 = \frac{\phi_{0,x}}{\phi_0} = \lambda_0 \tanh(\lambda_0 x - 4\lambda_0^3 t)$$

and also $\widetilde{K} = K(\Sigma) = K(\sigma_0) = K(e^{\lambda_0 x - 4\lambda_0^3 t})$. The solutions of (2.22) for $u = 0$ are $\phi_1 = \cosh(\lambda x - 4\lambda^3 t)$ and $\phi_2 = \sinh(\lambda x - 4\lambda^3 t)$ for $\lambda \neq \pm\lambda_0$. So, for $\lambda \neq \pm\lambda_0$, we have

$$L = K(e^{\lambda x - 4\lambda^3 t}).$$

Thus, $L \cap \widetilde{K} = K$, since $e^{\lambda_0 x - 4\lambda_0^3 t}$ and $e^{\lambda x - 4\lambda^3 t}$ are rationally independent when $\lambda_0 \neq \pm\lambda$.

Let G be the Galois group of system (2.24) and $\widetilde{G}$ the Galois group of the Darboux transformed system (2.31) for $\lambda \neq \pm\lambda_0$. Therefore, by Corollary 2.4, we have that

$$G \simeq \widetilde{G}, \tag{2.32}$$

and both Galois groups are isomorphic to the multiplicative group G_m, i.e.,

$$\left\{ \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix} : \alpha \in \mathbf{C}^* \right\}.$$

Now, take $\lambda_1 \neq \pm\lambda_0$, $\lambda_1 \neq 0$ in (2.22) for $u = 0$. Since $\phi_1 = \sinh(\lambda_1 x - 4\lambda_1^3 t)$ is a particular solution of such equation, then

$$\psi_{1,1} = \begin{pmatrix} \lambda_1\phi_1 + \phi_{1,x} \\ -\phi_1 \end{pmatrix} \qquad \text{and} \qquad \psi_{1,2} = \begin{pmatrix} -\lambda_1\phi_1 + \phi_{1,x} \\ -\phi_1 \end{pmatrix},$$

are solutions of (2.24) for $\lambda = \lambda_1$ and $\lambda = -\lambda_1$ respectively. Thus,

$$\widetilde{\psi}_{1,1} = D|_{\lambda=\lambda_1}\psi_{1,1} = \begin{pmatrix} (\lambda_1^2 - \lambda_1\sigma_0 + \sigma_0^2 - \lambda_0^2)\phi_1 + (\lambda_1 - \sigma_0)\phi_{1,x} \\ \sigma_0\phi_1 - \phi_{1,x} \end{pmatrix},$$

$$\widetilde{\psi}_{1,2} = D|_{\lambda=-\lambda_1}\psi_{1,2} = \begin{pmatrix} (\lambda_1^2 + \lambda_1\sigma_0 + \sigma_0^2 - \lambda_0^2)\phi_1 + (-\lambda_1 - \sigma_0)\phi_{1,x} \\ \sigma_0\phi_1 - \phi_{1,x} \end{pmatrix}$$

are particular solutions of (2.31) for $\lambda = \lambda_1$ and $\lambda = -\lambda_1$ respectively. Now, we can construct the Darboux transformation $\widetilde{D} = J(\lambda I - \widetilde{\Sigma})$, where $\widetilde{\Sigma} = \Psi_1\Lambda_1\Psi_1^{-1}$ for

$$\Lambda_1 = \begin{pmatrix} \lambda_1 & 0 \\ 0 & -\lambda_1 \end{pmatrix} \qquad \text{and} \qquad \Psi_1 = \begin{pmatrix} \widetilde{\psi}_{1,1} & \widetilde{\psi}_{1,2} \end{pmatrix}.$$

Since

$$\begin{aligned} \widetilde{\widetilde{P}} &= J(\widetilde{P} + [J, \widetilde{\Sigma}])J^{-1} = P + [J, \Sigma - \widetilde{\Sigma}] \\ &= \begin{pmatrix} 0 & 2(\lambda_1^2 - \lambda_0^2)\dfrac{(\lambda_1^2 - \lambda_0^2 + \sigma_0^2)\phi_1^2 - \phi_{1,x}^2}{-(\phi_{1,x} - \sigma_0\phi_1)^2} \\ -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -\widetilde{\widetilde{u}} \\ -1 & 0 \end{pmatrix} \end{aligned}$$

by Proposition 2.7 part (4), then:

$$\begin{aligned} \widetilde{\widetilde{u}} &= 2(\lambda_1^2 - \lambda_0^2)\frac{(\lambda_1^2 - \lambda_0^2 + \sigma_0^2)\phi_1^2 - \phi_{1,x}^2}{(\phi_{1,x} - \sigma_0\phi_1)^2} \\ &= \frac{-2(\lambda_1^2 - \lambda_0^2)[\lambda_1^2\cosh^2(z_0) + \lambda_0^2\sinh^2(z_1)]}{[\lambda_1\cosh(z_0)\cosh(z_1) - \lambda_0\sinh(z_0)\sinh(z_1)]^2}, \end{aligned} \tag{2.33}$$

where $z_i = \lambda_i x - 4\lambda_i^3 t$, $i = 0, 1$, is the two soliton solution of the KdV equation.

We point out that we can rewrite the expression (2.33) as:

$$\tilde{\tilde{u}} = u[2] = u - 2\partial_x^2 \ln W(\phi_0, \phi_1) = -2\partial_x^2 \ln W(\phi_0, \phi_1),$$

where $W(\phi_0, \phi_1)$ is the Wronskian, hence we get the Darboux–Crum transformation, as it is stated in [25] (see also [67]).

As we will show in next section, we can iterate the above procedure and obtain the d-soliton. In each iteration the Galois groups remain invariant, i.e., $G_i \simeq G_{i+1}$, $i = 1, \ldots, d-1$, because of Theorem 2.8, and isomorphim (2.32).

Remark 2.13. For $t = 0$ and $\lambda_i = (i+1)\lambda_0$, $i = 1, \ldots, d-1$, function $u[d]$ becomes:

$$u[d] = \frac{-d(d+1)\lambda_0^2}{\cosh^2(\lambda_0 x)}.$$

Rosen and Morse found that the Schrödinger equation with the above potential is solvable in closed form (in fact they studied a more general potential). A galoisian interpretation of this fact is given in [68].

2.5 Equivalence between Darboux–Crum transformations and decomposable matrix Darboux transformations

Now, we show that matrix Darboux transformations for the AKNS system (2.23) associated to the Schrödinger equation are equivalent to Darboux–Crum transformations for the Schrödinger equation as presented in Section 1.3. We explicitly express the relation between them.

Let ϕ be a solution of Schrödinger equation (2.22) and consider, as in previous section, $E + \lambda^2 = 0$. Recall that a Darboux–Crum transformation of degree n for ϕ is given by expression (1.16):

$$\phi[d] = \mathcal{D}[d]\phi.$$

We notice that this expression can be rewritten as

$$\phi[d] = Q_1\phi + Q_2\phi_x, \tag{2.34}$$

where Q_1, Q_2 are differential polynomials in $u, (u+\lambda^2), s_1, \ldots, s_d$.

Let us consider the $2d \times 2d$ matrix

$$\Gamma_d = \begin{pmatrix} \vec{\psi}_1 & \vec{\psi}_1^* & \cdots & \vec{\psi}_d & \vec{\psi}_d^* \\ \lambda_1\vec{\psi}_1 & (-\lambda_1)\vec{\psi}_1^* & \cdots & (\lambda_d)\vec{\psi}_d & (-\lambda_d)\vec{\psi}_d^* \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \lambda_1^{d-1}\vec{\psi}_1 & (-\lambda_1)^{d-1}\vec{\psi}_1^* & \cdots & (\lambda_d)^{d-1}\vec{\psi}_d & (-\lambda_d)^{d-1}\vec{\psi}_d^* \end{pmatrix},$$

where

$$\vec{\psi}_i = \begin{pmatrix} \lambda_i\phi_i + \phi_{i,x} \\ -\phi_i \end{pmatrix} \qquad \text{and} \qquad \vec{\psi}_i^* = \begin{pmatrix} -\lambda_i\phi_i + \phi_{i,x} \\ -\phi_i \end{pmatrix}, \qquad \text{for } \; i = 1, \ldots, d.$$

We observe that $\det \Gamma_d \neq 0$ since, by Remark 2.12, $\vec{\psi}_i$ is a fundamental solution of (2.23) for $\lambda = \lambda_i$ and $\vec{\psi}_i^*$ is a fundamental solution of (2.23) for $\lambda = -\lambda_i$, so setting $\Psi_i = (\vec{\psi}_i \quad \vec{\psi}_i^*)$ and $\Lambda = \operatorname{diag}(\lambda_i, -\lambda_i)$ we can rewrite Γ_d as:

$$\widetilde{\Gamma}_d = \begin{pmatrix} \Psi_1 & \cdots & \Psi_d \\ \Psi_1\Lambda_1 & \cdots & \Psi_d\Lambda_d \\ \vdots & \ddots & \vdots \\ \Psi_1\Lambda_1^{d-1} & \cdots & \Psi_d\Lambda_d^{d-1} \end{pmatrix},$$

which is an invertible matrix.

By Proposition 2.7 part (1), if

$$D_{\lambda_i}[d] \cdot \vec{\psi}_i = 0 \qquad \text{and} \qquad D_{-\lambda_i}[d] \cdot \vec{\psi}_i^* = 0, \qquad \text{for } \; i = 1, \ldots, d,$$

then,

$$D_\lambda[d] := D(\vec{\psi}_1, \vec{\psi}_1^*, \ldots, \vec{\psi}_d, \vec{\psi}_d^*, \lambda) = \widetilde{D}_d \left(\sum_{j=0}^{d-1} D_j(x,t)\lambda^j + \lambda^d I \right)$$

is a Darboux matrix of degree d for system (2.23). Moreover, since $\det \Gamma_{d-1} \neq 0$, we have

$$D_\lambda[d] = D_1 \cdot D_\lambda[d-1]$$

with $D_\lambda[d-1] = D(\vec{\psi}_1, \vec{\psi}_1^*, \ldots, \vec{\psi}_{d-1}, \vec{\psi}_{d-1}^*, \lambda)$ a Darboux matrix of degree $d-1$ and $D_1 = D_\lambda[1] = D(\vec{f}_1, \vec{f}_2, \lambda)$ a Darboux matrix of degree one where

$$\vec{f}_1 = D_{\lambda_d}[d-1] \cdot \vec{\psi}_d \qquad \text{and} \qquad \vec{f}_2 = D_{-\lambda_d}[d-1] \cdot \vec{\psi}_d^*.$$

Let $(\mathfrak{s}_d)$ be the corresponding system to the Darboux transform $D_\lambda[d]$. Because of Proposition 2.7 we have that the systems (2.23) and $(\mathfrak{s}_d)$ satisfy

$$\widetilde{P} = \widetilde{D}_d(P - [J, D_{d-1}])\widetilde{D}_d^{-1}. \tag{2.35}$$

With the same notations as above, we have the following results:

Proposition 2.14. *Let us take $\lambda_i \in \mathbf{C}^*$, $\lambda_i \neq \lambda_j$ if $i \neq j$, and $\phi_i \neq 0$ solutions of (2.22) for $E_i = -\lambda_i^2$, for $i = 1, \ldots, d$. The Darboux matrix of degree d, say $D_\lambda[d]$, satisfies:*

$$D_\lambda[d] \cdot \Phi = \Phi[d] = \begin{pmatrix} \lambda\phi[d] + \phi[d]_x \\ -\phi[d] \end{pmatrix},$$

where $\Phi = \begin{pmatrix} \lambda\phi + \phi_x \\ -\phi \end{pmatrix}$ is a general solution of (2.23) and ϕ is the general solution of (2.22).

Before starting the proof we make the following remark.

Remark 2.15. If we apply a Darboux–Crum transformation of order one to $\phi[d]$ we must obtain $\phi[d+1] = \mathcal{D}[d+1]\phi$, so:

$$\phi[d+1] = \phi[d]_x - \frac{\phi[d]_{d+1,x}}{\phi[d]_{d+1}}\phi[d] = \phi[d]_x - \sigma_{d+1}\phi[d], \tag{2.36}$$

where $\phi[d]_{d+1}$ is a solution of (1.14) for E_{d+1}.

Proof. We prove it by induction on the degree d. If $d = 1$, $D_\lambda[1]\Phi = \Phi[1]$ has the form:

$$\Phi[1] = \begin{pmatrix} (\lambda^2 - \lambda\sigma_0 + \sigma_0^2 - \lambda_0^2)\phi + (\lambda - \sigma_0)\phi_x \\ \sigma_0\phi - \phi_x \end{pmatrix},$$

by (2.27). Since $\phi[1] = \phi_x - \sigma_0\phi$ it follows that

$$\Psi[1] = \begin{pmatrix} \lambda\phi[1] + \phi[1]_x \\ -\phi[1] \end{pmatrix}.$$

Assume it holds for d: let

$$D_\lambda[d] = \begin{pmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{pmatrix} \tag{2.37}$$

be the Darboux matrix of degree d, then we have that

$$D_\lambda[d]\Phi = \Phi[d] = \begin{pmatrix} (\lambda d_{11} - d_{12})\phi + \phi_x d_{11} \\ (\lambda d_{21} - d_{22})\phi + \phi_x d_{21} \end{pmatrix} = \begin{pmatrix} \lambda\phi[d] + \phi[d]_x \\ -\phi[d] \end{pmatrix}. \tag{2.38}$$

Now, let us see it holds for $d+1$. Let $D_\lambda[d+1]$ be the Darboux matrix of degree $d+1$. As we have seen we can decompose it as a product of a Darboux matrix $D_\lambda[d]$ of degree d of the form (2.37) and a Darboux matrix of degree one of the form:

$$D_\lambda[1] = \begin{pmatrix} \lambda & 0 \\ 0 & -\lambda \end{pmatrix} - \begin{pmatrix} \sigma_{d+1} & \sigma_{d+1}^2 - \lambda_{d+1}^2 \\ 1 & \sigma_{d+1} \end{pmatrix},$$

with $\sigma_{d+1} = \frac{\phi[d]_{d+1,x}}{\phi[d]_{d+1}}$, where $\phi[d]_{d+1} \neq 0$ is a solution for $E_{d+1} = -\lambda_{d+1}^2$ with $E_{d+1} \neq E_i$, $1 \leq i \leq d$, of equation (1.14). Thus, we have:

$$\begin{aligned} D[d+1]_\lambda =& D_\lambda[1]D_\lambda[d] = \lambda \begin{pmatrix} d_{11} & d_{12} \\ -d_{21} & -d_{22} \end{pmatrix} - \\ & - \begin{pmatrix} \sigma_{d+1}d_{11} + d_{21}(\sigma_{d+1}^2 - \lambda_{d+1}^2) & \sigma_{d+1}d_{12} + d_{22}(\sigma_{d+1}^2 - \lambda_{d+1}^2) \\ d_{11} + \sigma_{d+1}d_{21} & d_{12} + \sigma_{d+1}d_{22} \end{pmatrix}. \end{aligned}$$

So,

$$
\begin{aligned}
D[d+1]_\lambda \Phi = \Phi[d+1] &= \lambda \begin{pmatrix} (\lambda d_{11} - d_{12})\phi + \phi_x d_{11} \\ (-\lambda d_{21} + d_{22})\phi - \phi_x d_{21} \end{pmatrix} + \\
&+ \begin{pmatrix} -\sigma_{d+1}[(\lambda d_{11} - d_{12})\phi + \phi_x d_{11}] - (\sigma_{d+1}^2 - \lambda_{d+1}^2)[(\lambda d_{21} - d_{22})\phi + \phi_x d_{21}] \\ -\sigma_{d+1}[(\lambda d_{21} - d_{22})\phi + \phi_x d_{21}] - (\lambda d_{11} - d_{12})\phi - \phi_x d_{11} \end{pmatrix} \\
&= \begin{pmatrix} \lambda(\lambda\phi[d] + \phi[d]_x) - \sigma_{d+1}(\lambda\phi[d] + \phi[d]_x) + (\sigma_{d+1}^2 - \lambda_{d+1}^2)\phi[d] \\ \lambda\phi[d] + \sigma_{d+1}\phi[d] - \lambda\phi[d] - \phi[d]_x \end{pmatrix} \\
&= \begin{pmatrix} \lambda\phi[d+1] + \phi[d+1]_x \\ -\phi[d+1] \end{pmatrix}
\end{aligned}
$$

by (2.36) and (2.38). □

Proposition 2.16. *The function* $\Phi[d] = \begin{pmatrix} \lambda\phi[d] + \phi[d]_x \\ -\phi[d] \end{pmatrix}$ *is a solution of the AKNS system*

$$
\begin{cases}
\widetilde{\Phi}_x &= \widetilde{U}\widetilde{\Phi} = \begin{pmatrix} \lambda & -\widetilde{u} \\ -1 & -\lambda \end{pmatrix} \widetilde{\Phi}\,, \\
\widetilde{\Phi}_t &= \widetilde{V}\widetilde{\Phi} = \begin{pmatrix} \lambda B[\widetilde{u}] - A[\widetilde{u}] & \lambda^2 B[\widetilde{u}] - 2\lambda A[\widetilde{u}] - C[\widetilde{u}] \\ -B[\widetilde{u}] & -\lambda B[\widetilde{u}] + A[\widetilde{u}] \end{pmatrix} \widetilde{\Phi}\,,
\end{cases}
\tag{2.39}
$$

where $E + \lambda^2 = 0$*, if and only if* $\widetilde{u} = u[d]$*, for* $u[d]$ *defined by* (1.13).

Proof. Since $\phi[d]$ satisfies equation (1.14), we have that $\Phi[d] = \begin{pmatrix} \phi[d] \\ \phi[d]_x \end{pmatrix}$ is a solution of the following flat system:

$$
\begin{cases}
\Phi_x &= U_d \Phi = \begin{pmatrix} 0 & 1 \\ u[d] + \lambda^2 & 0 \end{pmatrix} \Phi\,, \\
\Phi_t &= V_d \Phi = \begin{pmatrix} A[u[d]] & B[u[d]] \\ C[u[d]] & -A[u[d]] \end{pmatrix} \Phi\,.
\end{cases}
\tag{2.40}
$$

Now, we apply the gauge transformation $R = \begin{pmatrix} \lambda & 1 \\ -1 & 0 \end{pmatrix}$ to system (2.40) and we obtain that

$$\widetilde{\Phi}[d] = R \cdot \Phi[d] = \begin{pmatrix} \lambda\phi[d] + \phi[d]_x \\ -\phi[d] \end{pmatrix}$$

is a solution of the system:

$$
\begin{cases}
\widetilde{\Phi}_x &= \widetilde{U}_d \widetilde{\Phi} = R U_d R^{-1} \widetilde{\Phi} = \begin{pmatrix} \lambda & -u[d] \\ -1 & -\lambda \end{pmatrix} \widetilde{\Phi}\,, \\
\widetilde{\Phi}_t &= \widetilde{V}_d \widetilde{\Phi} = R V_d R^{-1} \widetilde{\Phi} \\
&= \begin{pmatrix} \lambda B[u[d]] - A[u[d]] & \lambda^2 B[u[d]] - 2\lambda A[u[d]] - C[u[d]] \\ -B[u[d]] & -\lambda B[u[d]] + A[u[d]] \end{pmatrix} \widetilde{\Phi}\,.
\end{cases}
\tag{2.41}
$$

So, if $\widetilde{u} = u[d]$, systems (2.39) and (2.41) are the same, thus $\Psi[d]$ is a solution of (2.39).

Conversely, if $\Psi[d]$ is a solution of (2.39), from the first equation we have that

$$\phi[d]_{xx} = (\widetilde{u} + \lambda^2)\phi[d],$$

hence $\widetilde{u} = u[d]$. □

Once we have the equivalence between the classical Darboux–Crum transformation and the matrix Darboux transformation, then using the construction found, for example, in [67], we can recover the d-soliton by means of decomposable Darboux matrices.

Chapter 3

KdV hierarchy and spectral curves

In this chapter we develop the Korteweg–de Vries hierarchy, onwards KdV hierarchy, for both the stationary case and the time dependent case. To do that, we use two equivalent formalisms: the differential operators approach and the integrable systems one.

The work exposed here is based on a joint work with Sonia Jiménez, Juan J. Morales and María Ángeles Zurro. The results that appear in this chapter are contained in the preprint [53].

3.1 Stationary KdV hierarchy

This section is devoted to the stationary KdV hierarchy, which we shall refer to as s-KdV hierarchy. We present two equivalent ways of obtaining it. The first approach will be via differential operators, in second place we will introduce the integrable systems one. Next, we will define the spectral curves for the s-KdV hierarchy and we will analyze how they behave under Darboux–Crum transformations.

Let K be a differential field with derivation ∂_x and field of constants $\mathbf{C}$ algebraically closed and of characteristic zero.

Let $E \in \mathbf{C}$ denote a parameter and $u = u(x) \in K$ be a fixed element of the differential field K.

We consider the differential recursive relations:

$$f_0 = 1, \qquad f_{j,x} = -\frac{1}{4} f_{j-1,xxx} + u f_{j-1,x} + \frac{1}{2} u_x f_{j-1}. \tag{3.1}$$

This recursion was first derived by Gel'fand and Dikii in [38]. There, they also provided an algorithm to compute $\partial_x^{-1}(f_{j,x})$. More recently, Olver proved that functions $f_{j,x}$ are total derivatives for every j and that functions f_j are differential polynomials in u, see [72] Theorem 5.31 (see also [71] Lemma 2.2 for a more detailed proof). For the first terms one finds

$$\begin{aligned}
&f_0 = 1, \qquad f_1 = \frac{1}{2}u + c_1, \qquad f_2 = -\frac{1}{8}u_{xx} + \frac{3}{8}u^2 + \frac{1}{2}c_1 u + c_2,\\
&f_3 = \frac{1}{32}u_{xxxx} - \frac{5}{16}uu_{xx} - \frac{5}{32}u_x^2 + \frac{5}{16}u^3 + c_1\left(-\frac{1}{8}u_{xx} + \frac{3}{8}u^2\right) + \frac{1}{2}c_2 u + c_3,
\end{aligned}$$

for some integration constants c_i.

Consider the Schrödinger operator $\mathcal{L} = -\partial_{xx} + u \in K[\partial_x]$ and the Schrödinger equation

$$(\mathcal{L} - E)\phi = (-\partial_{xx} + u - E)\phi = 0. \tag{3.2}$$

As in [40], we define the differential operators $\mathcal{A}_{2n+1} \in K[\partial_x]$ of order $2n+1$ by

$$\mathcal{A}_{2n+1} := \sum_{\ell=0}^{n}\left(f_{n-\ell}\partial_x - \frac{1}{2}f_{n-\ell,x}\right)\mathcal{L}^{\ell}. \tag{3.3}$$

The first few are

$$\mathcal{A}_1 = \partial_x, \qquad\qquad \mathcal{A}_3 = -\partial_x^3 + \frac{3}{2}u\partial_x + \frac{3}{4}u_x + c_1\partial_x.$$

Now, fix an n. Using differential recursion (3.1) we can explicitly compute the commutator of $\mathcal{L}$ and $\mathcal{A}_{2n+1}$:

$$[\mathcal{L}, \mathcal{A}_{2n+1}] = -2f_{n+1,x}(u). \tag{3.4}$$

When potential u is a solution of equation (3.4), both operators commute and we obtain the s-KdV$_n$ equation

$$\text{s-KdV}_n: \quad -2f_{n+1,x}(u) = 0. \tag{3.5}$$

Varying $n \in \mathbb{N}$, we get the *stationary KdV hierarchy*. The first members are:

$$\begin{aligned}
\text{s-KdV}_0:&\quad -u_x = 0,\\
\text{s-KdV}_1:&\quad \frac{1}{4}u_{xxx} - \frac{3}{2}uu_x - c_1u_x = 0,\\
\text{s-KdV}_2:&\quad -\frac{1}{16}u_{xxxxx} + \frac{5}{8}uu_{xxx} + \frac{5}{4}u_xu_{xx} - \frac{15}{8}u^2u_x + c_1\left(\frac{1}{4}u_{xx} - \frac{3}{2}uu_x\right)\\
&\quad - c_2u_x = 0.
\end{aligned}$$

Next, to explain the integrable systems approach we need to introduce time variables $\partial_{t_1}, \partial_{t_2}, \ldots, \partial_{t_m}$. For each level n of the s-KdV hierarchy, we will take the corresponding time t_n and study the corresponding time evolution ∂_{t_n}. Let K_n denote a differential field with compatible derivations ∂_x and ∂_{t_n}, in the variables x and t_r, for $n = 1, \ldots, m$, and with the same field of constants as K, $\mathbf{C}$ algebraically closed and of characteristic zero.

Let $E \in \mathbf{C}$ be a parameter and $u = u(x) \in K_n$ be a fixed time independent element of K_n.

Then, it is well known that the stationary s-KdV hierarchy can be constructed as zero curvature condition of the family of integrable systems (see [40] Chapter 1, Section 2):

$$\begin{cases} \Phi_x & = U\Phi = \begin{pmatrix} 0 & 1 \\ u-E & 0 \end{pmatrix} \Phi\,, \\ \\ \Phi_{t_n} & = V_n \Phi = \begin{pmatrix} G_n(u) & F_n(u) \\ -H_n(u) & -G_n(u) \end{pmatrix} \Phi\,, \end{cases} \tag{3.6}$$

where matrix U is the companion matrix of the Schrödinger operator and F_n, G_n and H_n are differential polynomials of the potential u defined by

$$F_n = \sum_{j=0}^{n} f_{n-j} E^j, \tag{3.7}$$

$$G_n = -\frac{F_{n,x}}{2}, \tag{3.8}$$

$$H_n = (E-u)F_n - G_{n,x} = (E-u)F_n + \frac{F_{n,xx}}{2}, \tag{3.9}$$

for $n = 0, 1, \ldots$. Observe that the degree in E of the matrix V_n and of the function H_n is $n+1$.

Remark 3.1. Notice that both matrix coefficients U and V are time independent since $u(x)$ is time independent and the functions f_j are differential polynomials in u. However, the solutions of system (3.6) will depend on t_n.

Now, fix a level n in the hierarchy and consider the corresponding system (3.6). Its zero curvature condition

$$-V_{n,x} + [U, V_n] = 0, \tag{3.10}$$

yields to the s-KdV$_n$ equation

$$\text{s-KdV}_n : \quad 0 = -\frac{1}{2} F_{n,xxx}(u) - 2(E-u)F_{n,x}(u) + u_x F_n(u). \tag{3.11}$$

Using expression (3.7), this equation can be also written as:

$$\begin{aligned} 0 &= -\frac{1}{2} F_{n,xxx} - 2(E-u)F_{n,x} + u_x F_n, \\ &= -\frac{1}{2} \sum_{j=0}^{n} E^j f_{n-j,xxx} - 2(E-u) \sum_{j=0}^{n} E^j f_{n-j,x} + u_x \sum_{j=0}^{n} E^j f_{n-j}, \\ &= \sum_{j=0}^{n} -2E^{j+1} f_{n-j,x} + E^j \left(-\frac{1}{2} f_{n-j,xxx} + 2u f_{n-j,x} + u_x f_{n-j} \right), \\ &= \sum_{j=0}^{n} -2(E^{j+1} f_{n-j,x} - E^j f_{n-j+1,x}) = 2 f_{n+1,x}. \end{aligned}$$

In this way, we obtain the same expression for the KdV_n equation as in the differential operators approach:

$$\text{s-KdV}_n: \quad 0 = 2f_{n+1,x}(u). \tag{3.12}$$

When u is a solution of s-KdV_n equation (3.12) for some n, we will say that it is a *s-KdV potential* or a *s-KdV_n potential* if it were necessary to specify the level of the hierarchy.

3.1.1 Spectral curves

In the same way as for the KdV hierarchy we find two approaches to obtain the spectral curves associated to the Schrödinger operator with s-KdV potentials, both of them equivalent: one with differential operators and other with integrable systems. The approach we are more interested in is the one with integrable systems. For this reason, we briefly introduce in first place the differential operators formalism.

We keep the notation of differential fields K and K_n of previous section.

Given two differential operators $\mathcal{L}, \mathcal{A} \in K[\partial_x]$, we can consider their commutator $[\mathcal{L}, \mathcal{A}]$. Burchnall and Chaundy proved in [19] that two operators commute, i.e., $[\mathcal{L}, \mathcal{A}] = 0$, if and only if they satisfy an algebraic identity with constant coefficients: $p(\mathcal{L}, \mathcal{A}) = 0$, where $p(y, z) \in \mathbf{C}[y, z]$. This algebraic relation is called the *spectral curve* of $\mathcal{L}$ and $\mathcal{A}$. Some years later, Previato showed in [76] that the spectral curve can be computed as the differential resultant $\partial_x \mathrm{Res}(\mathcal{L} - E, \mathcal{A} - \mu) = p(E, \mu) \in \mathbf{C}[E, \mu]$.

Now, take the Schrödinger operator $\mathcal{L}$ and the differential operators $\mathcal{A}_{2n+1}$ defined in (3.3). As we have shown, when u is a solution of the s-KdV_n equation both operators commute, thus, the spectral curve Γ_n of the pair $\{\mathcal{L}, \mathcal{A}_{2n+1}\}$ is defined by the polynomial in $\mathbf{C}[E, \mu]$

$$\Gamma_n : p_n(E, \mu) := \partial_x \mathrm{Res}(\mathcal{L} - E, \mathcal{A}_{2n+1} - \mu) = \mu^2 - R_{2n+1}(E) = 0,$$

where $R_{2n+1}(E)$ is a polynomial of degree $2n + 1$ in $\mathbf{C}[E]$ (as we will prove in Proposition 3.3).

In terms of integrable systems, when $u = u(x) \in K_n$ is a solution of the zero curvature condition (3.11), we obtain the spectral curve of system (3.6) for potential u as the characteristic polynomial of the matrix $iV_n(E, u)$:

$$\begin{aligned} \Gamma_n : \det(\mu I_2 - iV_n(E, u)) &= \mu^2 - \det(V_n(E, u)) = \mu^2 + G_n^2 - F_n H_n, \\ &= \mu^2 + \frac{F_{n,x}^2}{4} - (E - u)F_n^2 - \frac{F_n F_{n,xx}}{2}, \qquad (3.13) \\ &= \mu^2 - R_{2n+1}(E) = 0. \qquad (3.14) \end{aligned}$$

We denote by $p_n(E, \mu) = \mu^2 - R_{2n+1}(E)$ the equation that defines the spectral curve. We will use the following notation

$$R_{2n+1}(E) = \sum_{i=0}^{2n+1} C_i E^i, \tag{3.15}$$

where C_i are differential polynomials in u with constant coefficients.

Remark 3.2. Let ϕ_1 and ϕ_2 be the two fundamental solutions of Schrödinger equation (3.2) for the same value of the energy E_0. Then, by equation (3.14), we can find two values of μ, say μ_0 and $-\mu_0$, such that $(E_0, \mu) \in \Gamma_n$. We call the points (E_0, μ_0) and $(E_0, -\mu_0)$ the *corresponding points to* ϕ_1 *and* ϕ_2 *of the spectral curve* Γ_n. So, in this sense, each fundamental solution of the Schrödinger equation (3.2) corresponds to a point of the spectral curve.

With this matrix presentation it is easy to prove the following result:

Proposition 3.3 (Burchnall and Chaundy, [19])**.** *Let $u = u(x)$ be solution of equation* (3.12)*, we have that $p(\mu, E) = \mu^2 - R_{2n+1}(E) \in \mathbf{C}[\mu, E]$. Moreover, $R_{2n+1}(E)$ is a polynomial of degree $2n + 1$ in $C[E]$.*

Proof. It suffices to prove that $R_{2n+1}(E) \in \mathbf{C}[E]$ of degree $2n + 1$. By equation (3.13) we have that:

$$R_{2n+1}(E) = \det(V_n(E, u)) = -G_n^2 + F_n H_n = -\frac{F_{n,x}^2}{4} + (E - u)F_n^2 + \frac{F_n F_{n,xx}}{2}.$$

From this expression and (3.7), we know that $R_{2n+1}(E)$ is a polynomial of degree $2n + 1$ in $K[E]$. In order to see that the coefficients of $R_{2n+1}(E)$ are constant, we derive previous expression with respect to x:

$$(R_{2n+1}(E))_x = F_n \left(2(E - u)F_{n,x} - u_x F_n + \frac{F_{n,xxx}}{2} \right).$$

The part in brackets of this expression is precisely the zero curvature condition (3.11) of system (3.6), hence,

$$(R_{2n+1}(E))_x = 0.$$

So, we conclude that $R_{2n+1}(E) \in \mathbf{C}[E]$. $\square$

Remark 3.4. Proposition 3.3 can be also proved by means of Proposition 1.17 since the zero curvature condition (3.10) is equivalent to $V_n \in \mathcal{E}(U)$.

In fact, we can compute more precisely expressions for the coefficients of the spectral curve by means of their derivatives. For this, we compute $R_{2n+1}(E)$ in terms of the functions f_i using (3.7). We have:

$$\begin{aligned} R_{2n+1}(E) = -\frac{F_{n,x}^2}{4} + (E - u)F_n^2 + \frac{F_n F_{n,xx}}{2} = -\frac{1}{4} \left(\sum_{i=0}^{n} E^i f_{n-i,x} \right)^2 \\ - (u - E) \left(\sum_{i=0}^{n} E^i f_{n-i} \right)^2 + \frac{1}{2} \left(\sum_{i=0}^{n} E^i f_{n-i} \right) \left(\sum_{i=0}^{n} E^i f_{n-i,xx} \right) \\ = \sum_{i,j=0}^{n} E^{i+j} \left(\frac{-f_{n-i,x} f_{n-j,x}}{4} + \frac{f_{n-i} f_{n-j,xx}}{2} - u f_{n-i} f_{n-j} \right) \\ + \sum_{i,j=0}^{n} E^{i+j+1} f_{n-i} f_{n-j}. \end{aligned}$$

The coefficient of the term E^{2n+1} in this expression is $C_{2n+2} = f_0^2 = 1$. In order to compute the rest of coefficients C_i, $i = 0, 1, \dots, 2n$, we separate them in two sets: coefficients C_{2n-k}, for $k = 0, 1, \dots, n-1$, and coefficients C_{n-k}, for $k = 0, 1, \dots, n$. For the first set we find:

$$C_{2n-k} = \sum_{i+j=2n-k} \left(\frac{-f_{n-i,x} f_{n-j,x}}{4} + \frac{f_{n-i} f_{n-j,xx}}{2} - u f_{n-i} f_{n-j} \right) + \sum_{i+j=2n-k-1} f_{n-i} f_{n-j}.$$

The substitutions $j = 2n-k-i$ for $i = n-k, \dots, n$ in the first summation and $j = 2n-k-i-1$ for $i = n-k-1, \dots, n$, in the second one yield to

$$\begin{aligned} C_{2n-k} = & \sum_{i=n-k}^{n} \left(\frac{-f_{n-i,x} f_{k+i-n,x}}{4} + \frac{f_{n-i} f_{k+i-n,xx}}{2} - u f_{n-i} f_{k+i-n} \right) \\ & + \sum_{i=n-k-1}^{n} f_{n-i} f_{k+i+1-n}. \end{aligned}$$

Next, we derivate with respect to x this expression:

$$\begin{aligned} C_{2n-k,x} = & \sum_{i=n-k}^{n} \left(\frac{-f_{n-i,xx} f_{k+i-n,x} + f_{n-i,x} f_{k+i-n,xx} + 2 f_{n-i} f_{k+i-n,xxx}}{4} \right) \\ & - \sum_{i=n-k}^{n} (u_x f_{n-i} f_{k+i-n} + u f_{n-i,x} f_{k+i-n} + u f_{n-i} f_{k+i-n,x}) \\ & + \sum_{i=n-k-1}^{n} f_{n-i,x} f_{k+i+1-n} + f_{n-i} f_{k+i+1-n,x}. \end{aligned}$$

Notice that

$$\begin{aligned} \sum_{i=n-k}^{n} f_{n-i,xx} f_{k+i-n,x} &= \sum_{i=n-k}^{n} f_{n-i,x} f_{k+i-n,xx}, \\ \sum_{i=n-k}^{n} f_{n-i,x} f_{k+i-n} &= \sum_{i=n-k}^{n} f_{n-i} f_{k+i-n,x}, \\ \sum_{i=n-k-1}^{n} f_{n-i,x} f_{k+i+1-n} &= \sum_{i=n-k-1}^{n} f_{n-i} f_{k+i+1-n,x}. \end{aligned}$$

Using these equalities and expression (3.1), $C_{2n-k,x}$ simplifies to

$$\begin{aligned} C_{2n-k,x} &= \sum_{i=n-k}^{n} f_{n-i} \left(\frac{f_{k+i-n,xxx}}{2} - u_x f_{k+i-n} - 2u f_{k+i-n,x} \right) \\ &\quad + \sum_{i=n-k-1}^{n} 2 f_{n-i} f_{k+i+1-n,x} \\ &= \sum_{i=n-k}^{n} -2 f_{n-i} f_{k+i-n+1,x} + \sum_{i=n-k-1}^{n} 2 f_{n-i} f_{k+i+1-n,x} = 2 f_{k+1} f_{0,x} = 0, \end{aligned}$$

since $f_0 = 1$. Thus, $C_{2n-k} \in \mathbf{C}$ for $k = 0, 1, \ldots, n-1$.

By an analogue computation, we obtain that

$$C_{n-k,x} = -2 f_k f_{n+1,x}$$

for $k = 0, 1, \ldots, n$. Therefore, when the potential u is a solution of s-KdV$_n$ equation, by equation (3.12) we have that $2f_{n+1,x}(u) = 0$. Then, $C_{n-k,x} = 0$ and hence $C_{n-k} \in \mathbf{C}$ for $k = 0, 1, \ldots, n$, as stated in Proposition 3.3. Otherwise, when u is not a solution of s-KdV$_n$ equation, the coefficients of $R_{2n+1}(E)$ do not have to be constant and expression (3.13) is no longer a spectral curve.

In particular, we point out the expression for $C_{0,x}$:

$$C_{0,x} = -2 f_n f_{n+1,x}. \tag{3.16}$$

This formula together with Proposition 3.3 and recursion formula (3.1) easily implies the following result.

Corollary 3.5. *Let $\mu^2 - R_{2n+1}(E)$ be the spectral curve for potential u. If the degree of $R_{2n+1}(E)$ is $2n+1$ in E, then u is solution of a s-KdV$_n$ equation.*

Proof. Combining 3.3 and (3.16) we have

$$C_{0,x} = -2 f_n(u) f_{n+1,x}(u) = 0.$$

So, either $-2f_{n+1,x}(u) = 0$ and we get the result or $f_n = 0$. In this case, applying recursion formula (3.1) the result follows. $\square$

Next, in order to include in our study the point of infinity of the spectral curve, we consider the Zariski closure of $\Gamma_n \subset \mathbf{C}^2$, say $\overline{\Gamma}_n$, in the complex projective plane $\mathbb{P}^2$. Let $p(E, \mu) = \mu^2 - R_{2n+1}(E) = \mu^2 - \sum_{j=0}^{2n+1} C_j E^j = 0$ be an equation for Γ_n. Then, for $[E : \mu : \nu] \in \mathbb{P}^2$, an equation for $\overline{\Gamma}_n$ is

$$p_h(E, \mu, \nu) = \mu^2 \nu^{2n-1} - \widehat{R}_{2n+1}(E, \nu) = 0, \tag{3.17}$$

where $\widehat{R}_{2n+1}(E, \nu) = \nu^{2n+1} R_{2n+1}\left(\frac{E}{\nu}\right) = \sum_{j=0}^{2n+1} C_j \nu^{2n+1-j} E^j$ is an homogeneous polynomial of degree $2n+1$. Moreover, observe that the singular points of $\overline{\Gamma}_n$ are

$$\operatorname{Sing}(\overline{\Gamma}_n) = \{(E, 0) \ : \ E \ \text{ is a multiple root of } R_{2n+1}\} \cup \{P_\infty = [0 : 1 : 0]\}.$$

Also we find

$$\overline{\Gamma}_n \cap \{E = 0\} = \{[0 : \mu : \nu] \in \mathbb{P}^2 : \mu^2 \nu^{2n-1} = C_0 \nu^{2n+1}\}. \tag{3.18}$$

3.1.2 Extended Green's function

The purpose of the rest of Section 3.1 is to study how Darboux–Crum transformations affect the spectral curve. With this aim, we introduce the diagonal Green's function. The relation between the diagonal Green's function and the spectral curves will come evident in equation (3.27). Since we want to include in our analysis the point of infinity of the spectral curve, at some point we will also have to introduce a homogenized expression of the diagonal Green's function (see equation (3.28)) as we did for the spectral curve.

For the rest of this section, we work over the differential field K with derivation ∂_x and we take $u = u(x) \in K$.

Following [40], we define the diagonal Green's function on $\Gamma_n \times \mathbf{C}$ as

$$g(E, \mu, x) = \frac{\phi_1 \phi_2}{W(\phi_1, \phi_2)} \tag{3.19}$$

where ϕ_1 and ϕ_2 are two independent solutions of Schrödinger equation

$$(\mathcal{L} - E)\phi = (-\partial_{xx} + u - E)\phi = 0, \tag{3.20}$$

for the same value of E and $W(\phi_1, \phi_2)$ stands for their wronskian. We observe that $W(\phi_1, \phi_2) \in \mathbf{C}$ by Liouville's Formula 1.1.

Remark 3.6. The Green's function of $\mathcal{L}$, $G(E, x_1, x_2)$ for x_1 and x_2 real variables, is the integral kernel of the resolvent $(\mathcal{L} - E)^{-1}$ and is given by

$$G(E, x_1, x_2) = \begin{cases} \dfrac{\phi_1(E, x_1) \cdot \phi_2(E, x_2)}{W(\phi_1(E, x_1), \phi_2(E, x_1))} & \text{if } x_1 \geq x_2, \\ \dfrac{\phi_1(E, x_2) \cdot \phi_2(E, x_1)}{W(\phi_1(E, x_1), \phi_2(E, x_1))} & \text{if } x_1 \leq x_2. \end{cases}$$

The diagonal Green's function is obtained from this expression, obviously, by taking $x_1 = x_2 = x$.

The function $X = \phi_1 \phi_2$ and, consequently, the diagonal Green's function are solutions of the third order linear differential equation:

$$\phi_{xxx} = 2u_x \phi + 4(u - E)\phi_x. \tag{3.21}$$

This equation is the second symmetric power of the Schrödinger equation, whose solution space is generated by $\{\phi_1^2, 2\phi_1\phi_2, \phi_2^2\}$, as we saw in Subsection 1.2.3. Observe that this expression is the same as formula (3.11).

As far as we know, the relevance of the equation (3.21) for the KdV equation was considered for the first time by Gel'fand and Dikii in their fundamental paper about the asymptotic behaviour of the resolvent of the Schrödinger equation associated to the KdV equation [38].

For simplicity henceforward we will refer to the diagonal Green's function simply as *Green's function*.

Let

$$\sigma_+ = \sigma(E,\mu) = \frac{i\mu + F_{n,x}/2}{F_n}, \qquad \sigma_- = \sigma(E,-\mu) = \frac{-i\mu + F_{n,x}/2}{F_n} \tag{3.22}$$

be functions defined over the spectral curve, i.e., $(E,\mu) \in \Gamma_n$. We recall the following result.

Lemma 3.7 (Lemma 1.8 in [40])**.** *Let u be solution of s-KdV$_n$ equation* (3.12)*. Let ϕ_1 and ϕ_2 be solutions of Schrödinger equation* (3.20) *for this potential and with corresponding functions over the spectral curve σ_+ and σ_- defined by* (3.22)*. Then, σ_+ and σ_- are solutions of the Riccati type equation:*

$$\sigma^2 + \sigma_x = u - E. \tag{3.23}$$

Moreover, the following identities are satisfied:

$$\sigma_+ + \sigma_- = \frac{F_{n,x}}{F_n} = \frac{(\phi_1\phi_2)_x}{\phi_1\phi_2}, \tag{3.24}$$

$$\sigma_+ - \sigma_- = \frac{2i\mu}{F_n} = -\frac{W(\phi_1,\phi_2)}{\phi_1\phi_2}, \tag{3.25}$$

$$\sigma_+ \cdot \sigma_- = \frac{H_n}{F_n} = \frac{\phi_{1,x}\phi_{2,x}}{\phi_1\phi_2}, \tag{3.26}$$

where $W(\phi_1,\phi_2) = \phi_1\phi_{2,x} - \phi_{1,x}\phi_2$ denotes the wronskian of ϕ_1 and ϕ_2.

We remark that this lemma is essentially a reformulation of a classic result that goes back to Hermite when he was studying closed form solutions for Lamé equation ([43]). In [88] the authors call this approach the Lindeman–Stieljes theory but, as far as we know, this approach was used for the first time by Hermite, and then by others: Halphen, Brioschi, Crawford, Stieljes.... The method used that the product of solutions $X = \phi_1\phi_2$ is a solution of the second symmetric power of the Schrödinger equation, see (3.21). Then the relations (3.24)-(3.26) connect the solutions of the Riccati equation with that of the second symmetric power. The fact that there is a connection between the solutions of the second symmetric product and the Riccati equation of the Schrödinger equation is relevant for the differential Galois theory (recall Theorem 1.21). Furthermore it is interesting to point out that the solutions of the Lamé equation obtained by Hermite in [43], are associated to other algebro-geometric solutions of KdV equation, finite-gap solutions with regular spectral curves, see [71] and references therein.

By Lemma 3.7, the Green's function can be rewritten as

$$g(E,\mu,x) = \frac{iF_n(E,x)}{2\mu} = \frac{1}{\sigma_- - \sigma_+}, \qquad \text{for } (E,\mu) \in \Gamma_n. \tag{3.27}$$

Observe that g is well defined whenever $\mu \neq 0$, i.e., for energy levels E such that $R_{2n+1}(E) \neq 0$. Using expression (3.27) we get the following differential relation for the function g:

$$\frac{1}{2} g g_{xx} - (u-E)g^2 - \frac{1}{4} g_x^2 = -\frac{1}{4},$$

since $g_x = (\sigma_+ + \sigma_-)g$ and $g_{xx} = 2(u - E + \sigma_+\sigma_-)g$.

Next, we define an extension of g on $\overline{\Gamma}_n \times \mathbf{C}_x$ as

$$g_h(E,\mu,\nu,x) = \frac{i\nu^n F_n(E/\nu,x)}{2\mu\nu^{n-1}}, \quad \text{for } [E:\mu:\nu] \in \overline{\Gamma}_n \setminus \{\,\mu\nu = 0\,\}. \tag{3.28}$$

We call g_h the *homogenized Green's function*. This function is well defined and extends g since, for $(E,\mu) \in \Gamma_n$ and $[E:\mu:\nu] \in \overline{\Gamma}_n$, we find

$$g_h(aE,a\mu,a\nu,x) = g_h(E,\mu,\nu,x) \qquad \text{and} \qquad g_h(E,\mu,1,x) = g(E,\mu,x),$$

for any $a \in \mathbf{C}$, $a \neq 0$. Moreover, we have that

$$\widehat{F}_n(E,\nu,x) := \nu^n F_n(E/\nu,x) = \sum_{j=0}^{n} f_{n-j}\nu^{n-j}E^j \tag{3.29}$$

is an homogeneous polynomial in E and ν of degree n and then

$$g_h(E,\mu,\nu,x) = \frac{i\widehat{F}_n(E,\nu,x)}{2\mu\nu^{n-1}}, \quad \text{for } [E:\mu:\nu] \in \overline{\Gamma}_n. \tag{3.30}$$

Using this notation we get the following formula for $[E:\mu:\nu] \in \overline{\Gamma}_n$:

$$\mu^2\nu^{2n-2} = \nu^{2n}R_{2n+1}(E/\nu) = \frac{\nu\widehat{F}_n\widehat{F}_{n,xx}}{2} - (u - E/\nu)\widehat{F}_n^2 - \frac{\nu^2\widehat{F}_{n,x}^2}{4}, \tag{3.31}$$

where

$$\widehat{F}_{n,x} = \nu^{n-1}F_{n,x}(E/\nu) \qquad \text{and} \qquad \widehat{F}_{n,xx} = \nu^{n-1}F_{n,xx}(E/\nu) \tag{3.32}$$

are homogeneous polynomials in E and ν of degree $n-1$.

Finally, we define the extensions of σ_+ and σ_- on $\overline{\Gamma}_n \times \mathbf{C}_x$ as

$$(\sigma_+)_h = \frac{i\mu\nu^{n-1} + \nu\widehat{F}_{n,x}/2}{\widehat{F}_n} \qquad \text{and} \qquad (\sigma_-)_h = \frac{-i\mu\nu^{n-1} + \nu\widehat{F}_{n,x}/2}{\widehat{F}_n}\,. \tag{3.33}$$

Notice that the functions $(\sigma_+)_h$ and $(\sigma_-)_h$ satisfy the Riccati type relation

$$(\sigma_\pm^2 + (\sigma_\pm)_x)_h = ((\sigma_\pm)_h)^2 + ((\sigma_\pm)_x)_h = u - E/\nu,$$

where

$$((\sigma_\pm)_x)_h = \frac{\nu\widehat{F}_n\widehat{F}_{n,xx}/2 - i\mu\nu^n\widehat{F}_{n,x} - \nu^2\widehat{F}_{n,x}^2/2}{\widehat{F}_n^2},$$

for $\widehat{F}_{n,x}$ and $\widehat{F}_{n,xx}$ defined by (3.32). Moreover we have that the function

$$g_h = \frac{i\widehat{F}_n(E,\nu,x)}{2\mu\nu^{n-1}} = \frac{1}{(\sigma_-)_h - (\sigma_+)_h} \tag{3.34}$$

verifies the non linear differential equation

$$\frac{1}{2}g_h(g_{xx})_h - (u - E/\nu)g_h^2 - \frac{1}{4}(g_x^2)_h = -\frac{1}{4},$$

since $(g_x)_h = ((\sigma_-)_h + (\sigma_+)_h) \cdot g_h$ and $(g_{xx})_h = 2(u - E/\nu + (\sigma_+)_h(\sigma_-)_h) \cdot g_h$.

3.1.2.1 Transformed Green's functions

Now, we analyze how Darboux–Crum transformations change Green's functions g and g_h. For that, we will use solutions of the Riccati type equation (3.23) as a esential tool.

Let u be solution of s-KdV$_n$ equation (3.12). Let ϕ_1 and ϕ_2 be solutions of Schrödinger equation (3.20) for this potential and energy level E. The corresponding points of the spectral curve for these functions are, respectively, (E, μ) and $(E, -\mu)$ (see Remark 3.2). Next we consider ϕ_0 a solution of the Schrödinger equation for u and E_0, with $E_0 \neq E$, and choose as corresponding point of the spectral curve (E_0, μ_0). Recall that in Section 1.3 we see that when applying a Darboux–Crum transformation with ϕ_0 to u, ϕ_1 and ϕ_2, we get

$$DT(\phi_0)u = u - 2\sigma_{0,x}, \quad DT(\phi_0)\phi_1 = \phi_{1,x} - \sigma_0\phi_1, \quad DT(\phi_0)\phi_2 = \phi_{2,x} - \sigma_0\phi_2,$$

where $\sigma_0 = (\log \phi_0)_x$ is a solution of the Riccati equation $\sigma^2 + \sigma_x = u - E_0$. By Lemma 3.7, the function σ^0 equals

$$\sigma^0 = \sigma(E_0, \mu_0) = \frac{i\mu_0 + F^0_{n,x}/2}{F^0_n}, \tag{3.35}$$

where $F^0_n = F_n(E_0)$, is a solution of the same Riccati equation as σ_0 for the same energy value E_0. Thus, we conclude that we can perform a Darboux–Crum transformation using σ^0 instead of σ_0. The transformed functions

$$\widetilde{\phi}_1 = \phi_{1,x} - \sigma^0\phi_1 \qquad \text{and} \qquad \widetilde{\phi}_2 = \phi_{2,x} - \sigma^0\phi_2 \tag{3.36}$$

are solutions of the Schrödinger equation for potential

$$\widetilde{u} = u - 2\sigma^0_x.$$

Now, we take the functions $\sigma_1 = (\log \phi_1)_x$ and $\sigma_2 = (\log \phi_2)_x$, which are solutions of the Riccati equation (3.23) for $E \neq E_0$. Then, by equations (3.24)-(3.26), we get the equalities

$$\sigma_+ - \sigma_- = \frac{2i\mu}{F_n} = -\frac{W(\phi_1, \phi_2)}{\phi_1\phi_2} = \frac{\phi_{1,x}}{\phi_1} - \frac{\phi_{2,x}}{\phi_2} = \sigma_1 - \sigma_2, \tag{3.37}$$

$$\sigma_+ + \sigma_- = \frac{F_{n,x}}{F_n} = \frac{\phi_1\phi_{2,x} + \phi_{1,x}\phi_2}{\phi_1\phi_2} = \frac{\phi_{1,x}}{\phi_1} + \frac{\phi_{2,x}}{\phi_2} = \sigma_1 + \sigma_2, \tag{3.38}$$

$$\sigma_+ \cdot \sigma_- = \frac{\phi_{1,x}\phi_{2,x}}{\phi_1\phi_2} = \frac{\phi_{1,x}}{\phi_1} \cdot \frac{\phi_{2,x}}{\phi_2} = \sigma_1 \cdot \sigma_2. \tag{3.39}$$

Next, we define the transformed Green's function

$$\widetilde{g}(E, \mu, x) = \frac{\widetilde{\phi}_1\widetilde{\phi}_2}{W(\widetilde{\phi}_1, \widetilde{\phi}_2)} = \frac{(\sigma_1 - \sigma^0)(\sigma_2 - \sigma^0)}{(E - E_0)} \cdot \frac{\phi_1\phi_2}{W(\phi_1, \phi_2)}, \tag{3.40}$$

by (3.36). Relations (3.37)-(3.39) link the Green's function g and the transformed Green's function $\widetilde{g}$ as follows:

$$\widetilde{g}(E,\mu,x) = \frac{(\sigma_1-\sigma^0)(\sigma_2-\sigma^0)}{(E-E_0)}\cdot\frac{\phi_1\phi_2}{W(\phi_1,\phi_2)} = \frac{(\sigma_+-\sigma^0)(\sigma_--\sigma^0)}{(E-E_0)}\cdot g(E,\mu,x)\,.$$

Hence we obtain a rational presentation of $\widetilde{g}$ as a consequence of the formulas (3.22) and (3.35). We write this formula in (3.41).

Proposition 3.8. *The Green's function associated to the transformed Schrödinger operator explicitly reads:*

$$\widetilde{g}(E,\mu,x) = \frac{i\left(\mu^2(F_n^0)^2-\mu_0^2F_n^2-i\mu_0F_n(F_n^0F_{n,x}-F_{n,x}^0F_n)+\frac{(F_n^0F_{n,x}-F_{n,x}^0F_n)^2}{4}\right)}{2\mu(E-E_0)F_n(F_n^0)^2}. \tag{3.41}$$

The following result states the structure of $\widetilde{g}$:

Proposition 3.9 (Lemma G.1 in [40])**.** *Let u be a solution of s-KdV$_n$ equation, let (E_0,μ_0) and (E,μ) be two different points of Γ_n. Then the Green's function explicitly reads:*

$$\widetilde{g}(E,\mu,x) = \frac{(\sigma_+-\sigma^0)(\sigma_--\sigma^0)}{(E-E_0)}\cdot\frac{iF_n}{2\mu} = \frac{i\widetilde{F}_{\widetilde{n}}(E,x)}{2\widetilde{\mu}}, \tag{3.42}$$

where $\widetilde{F}_{\widetilde{n}}$ is a polynomial in E of degree $\widetilde{n}$ and $\widetilde{\mu}$ is such that $\Gamma_{\widetilde{n}}:\widetilde{\mu}^2-\widetilde{R}_{2\widetilde{n}+1}=0$ for some polynomial $\widetilde{R}_{2\widetilde{n}+1}(E)$ of degree $2\widetilde{n}+1$, with $0\le\widetilde{n}\le n+1$.

Next, for the homogenized Green's function, choose the point of the spectral curve $[E_0:\mu_0:\nu_0]$. We define the extension of σ^0 on $\overline{\Gamma}_n\times\mathbf{C}_x$ as

$$(\sigma^0)_h(E_0,\mu_0,\nu_0) = \frac{i\mu_0\nu_0^{n-1}+\nu_0\widehat{F}_{n,x}^0/2}{\widehat{F}_n^0}, \tag{3.43}$$

where $\widehat{F}_n^0=\widehat{F}_n(E_0,\nu_0,x)$ for $\widehat{F}_n$ defined by (3.29) and $\widehat{F}_{n,x}^0=\widehat{F}_{n,x}(E_0,\nu_0,x)$, for $\widehat{F}_{n,x}$ defined in (3.32). Notice that, when $\nu_0=0$, function $(\sigma^0)_h$ vanishes. Hence, whenever $\nu_0=0$, we define

$$(\sigma^0)_h(E_0,\mu_0,0) := 0\,, \qquad \text{for } [E_0:\mu_0:0]\in\overline{\Gamma}_n.$$

Using above notation we have the following results.

Proposition 3.10. *Let assume $C_0=R_{2n+1}(0)\neq 0$. For $E_0=0$ and $\mu_0\neq 0$, the homogenized Green's function associated to the transformed Green's function $\widetilde{g}$ for $-\partial_{xx}+\widetilde{u}-E$ explicitly reads:*

$$\begin{aligned}(\widetilde{g})_h(E,\mu,\nu,x) = {}& \frac{i\left(\frac{\nu^2\widehat{F}_{n,xx}}{2}+(E-\nu u)\widehat{F}_n+\frac{\nu f_{n,x}^2\widehat{F}_n}{4f_n^2}-\frac{\nu^2 f_{n,x}\widehat{F}_{n,x}}{2f_n}-\frac{\nu C_0\widehat{F}_n}{f_n^2}\right)}{2\mu E\nu^{n-1}} \\ &+\frac{C_0\nu_0(\nu f_n\widehat{F}_{n,x}-f_{n,x}\widehat{F}_n)}{2\mu E\nu^{n-2}\mu_0 f_n^2},\end{aligned} \tag{3.44}$$

where $\widehat{F}_n(E,\nu,x)$ is defined by (3.29) and $\widehat{F}_{n,x}(E,\nu,x)$, $\widehat{F}_{n,xx}(E,\nu,x)$ are defined by (3.32).

Remark 3.11. Formula

$$\frac{\nu^2\widehat{F}_{n,xx}}{2}+(E-\nu u)\widehat{F}_n+\frac{\nu f_{n,x}^2\widehat{F}_n}{4f_n^2}-\frac{\nu^2 f_{n,x}\widehat{F}_{n,x}}{2f_n}-\frac{\nu C_0\widehat{F}_n}{f_n^2}$$

is an homogeneous polynomial in E and ν of degree $n+1$.

Proof. First, consider the transformed Green's function $\widetilde{g}$ given by (3.41). Then, the homogenized Green's function is obtained by the homogenization process as

$$\begin{aligned}(\widetilde{g})_h(E,\mu,\nu,x)&=\left(\frac{(\sigma_+-\sigma^0)(\sigma_--\sigma^0)}{(E-E_0)}\cdot\frac{iF_n}{2\mu}\right)_h\\&=\frac{i\left(\mu^2\nu^{2n-2}(\widehat{F}_n^0)^2+\frac{(\nu\widehat{F}_n^0\widehat{F}_{n,x}-\nu_0\widehat{F}_{n,x}^0\widehat{F}_n)^2}{4}\right)}{2\mu\nu^{n-1}(E/\nu-E_0/\nu_0)\widehat{F}_n(\widehat{F}_n^0)^2}\\&\quad-\frac{i\mu_0\nu_0^{n-1}\left(\mu_0\nu_0^{n-1}\widehat{F}_n^2+i\widehat{F}_n(\nu\widehat{F}_n^0\widehat{F}_{n,x}-\nu_0\widehat{F}_{n,x}^0\widehat{F}_n)\right)}{2\mu\nu^{n-1}(E/\nu-E_0/\nu_0)\widehat{F}_n(\widehat{F}_n^0)^2},\end{aligned}$$

where $\widehat{F}_n(E,\nu,x)$ is defined by (3.29), $\widehat{F}_{n,x}(E,\nu,x)$ is defined in (3.32), $\widehat{F}_n^0=\widehat{F}_n(E_0,\nu_0,x)$ and $\widehat{F}_{n,x}^0=\widehat{F}_{n,x}(E_0,\nu_0,x)$. In particular, for $E_0=0$, we get:

$$\begin{aligned}(\widetilde{g})_h(E,\mu,\nu,x)&=\frac{i\left(\mu^2\nu^{2n-2}f_n^2+\frac{(\nu f_n\widehat{F}_{n,x}-f_{n,x}\widehat{F}_n)^2}{4}\right)}{2\mu E\nu^{n-2}\widehat{F}_nf_n^2}-\frac{i\mu_0^2\widehat{F}_n}{2\mu E\nu^{n-2}\nu_0^2f_n^2}\\&\quad+\frac{\mu_0(\nu f_n\widehat{F}_{n,x}-f_{n,x}\widehat{F}_n)}{2\mu E\nu^{n-2}\nu_0f_n^2},\end{aligned}$$

since $\widehat{F}_n(0,\nu_0,x)=\nu_0^nf_n$ and $\widehat{F}_{n,x}(0,\nu_0,x)=\nu_0^{n-1}f_{n,x}$. Considering (3.31) we get the following expression

$$\begin{aligned}(\widetilde{g})_h(E,\mu,\nu,x)&=\frac{i\left(\frac{\nu^2\widehat{F}_{n,xx}}{2}+(E-\nu u)\widehat{F}_n+\frac{\nu f_{n,x}^2\widehat{F}_n}{4f_n^2}-\frac{\nu^2f_{n,x}\widehat{F}_{n,x}}{2f_n}\right)}{2\mu E\nu^{n-1}}-\frac{i\mu_0^2\widehat{F}_n}{2\mu E\nu^{n-2}\nu_0^2f_n^2}\\&\quad+\frac{\mu_0(\nu f_n\widehat{F}_{n,x}-f_{n,x}\widehat{F}_n)}{2\mu E\nu^{n-2}\nu_0f_n^2}.\end{aligned}$$

Moreover, by (3.18) we have that $\mu_0^2=C_0\nu_0^2$, and then

$$\begin{aligned}(\widetilde{g})_h(E,\mu,\nu,x)&=\frac{i\left(\frac{\nu^2\widehat{F}_{n,xx}}{2}+(E-\nu u)\widehat{F}_n+\frac{\nu f_{n,x}^2\widehat{F}_n}{4f_n^2}-\frac{\nu^2f_{n,x}\widehat{F}_{n,x}}{2f_n}\right)}{2\mu E\nu^{n-1}}-\frac{iC_0\widehat{F}_n}{2\mu E\nu^{n-2}f_n^2}\\&\quad+\frac{C_0\nu_0(\nu f_n\widehat{F}_{n,x}-f_{n,x}\widehat{F}_n)}{2\mu E\nu^{n-2}\mu_0f_n^2}.\end{aligned}$$

And then the result follows. □

Proposition 3.12. *Let assume $C_0 = R_{2n+1}(0) = 0$. For $E_0 = 0$ and $\mu_0 \neq 0$, the homogenized Green's function associated to the transformed Green's function $\widetilde{g}$ for $-\partial_{xx} + \widetilde{u} - E$ explicitly reads:*

$$(\widetilde{g})_h(E,\mu,\nu,x) = \frac{i\left(\frac{\nu^2 \widehat{F}_{n,xx}}{2} + (E-\nu u)\widehat{F}_n\right)}{2\mu E \nu^{n-1}}, \tag{3.45}$$

where $\widehat{F}_n(E,\nu,x)$ is defined by (3.29) *and $\widehat{F}_{n,xx}(E,\nu,x)$ is defined in* (3.32).

Remark 3.13. Formula

$$\frac{\nu^2 \widehat{F}_{n,xx}}{2} + (E-\nu u)\widehat{F}_n$$

is an homogeneous polynomial in E and ν of degree $n+1$.

Proof. When $C_0 = 0$ we have that $\nu_0 = 0$ by (3.18), since $\mu_0 \neq 0$. So, $(\sigma^0)_h = 0$. Hence, the homogenized Green's function in this case is:

$$\begin{aligned}(\widetilde{g})_h(E,\mu,\nu,x) &= \frac{(\sigma_+)_h(\sigma_-)_h}{E/\nu} \cdot \frac{i\widehat{F}_n}{2\mu\nu^{n-1}} = \frac{i\left(\frac{\mu^2\nu^{2n-2} + \nu^2 \widehat{F}_{n,x}^2/4}{\widehat{F}_n}\right)}{2\mu E \nu^{n-2}} \\ &= \frac{i\left(\frac{\nu^2 \widehat{F}_{n,xx}}{2} + (E-\nu u)\widehat{F}_n\right)}{2\mu E \nu^{n-1}},\end{aligned}$$

by (3.33) and (3.31). $\square$

3.1.3 Darboux–Crum transformations for the spectral curve

Finally, we present how Darboux–Crum transformations affect the spectral curve Γ_n and its closure $\overline{\Gamma}_n$. We observe that the action of the tranformation $DT(\phi_0)$ strongly depend on the type of point P in the spectral curve we use to construct ϕ_0 (recall Remark 3.2). In fact, if P is a regular point, the curve associated to the transformed potential is the same; in the other cases the new curve is a blowing-down or a blowing-up of Γ_n.

Theorem 3.14 (I)**.** *Let $(E_0,\mu_0) \in \Gamma_n$ and u be a solution of s-KdV$_n$ equation. Let ϕ_0 be a solution of Schrödinger equation for energy E_0 and potential u, i.e., $\phi_{0,xx} = (u-E_0)\phi_0$. Let $\widetilde{u} = u - 2(\log\phi_0)_{xx}$ be the Darboux–Crum transformation of u. Then, $\widetilde{u}$ is solution of a s-KdV$_{\widetilde{n}}$ equation for*

$$\widetilde{n} = \begin{cases} n & \text{if } (E_0,\mu_0) \text{ is a regular point of } \Gamma_n, \\ n-1 & \text{if } (E_0,\mu_0) \text{ is an affine singular point of } \Gamma_n. \end{cases}$$

Futhermore, the spectral curve associated to $\widetilde{u}$ is $\Gamma_{\widetilde{n}} : \widetilde{\mu}^2 - \widetilde{R}_{2\widetilde{n}+1} = 0$, with

$$\widetilde{R}_{2\widetilde{n}+1} = \begin{cases} R_{2n+1} & \text{if } (E_0,\mu_0) \text{ is a regular point of } \Gamma_n, \\ (E-E_0)^{-2}R_{2n+1} & \text{if } (E_0,\mu_0) \text{ is an affine singular point of } \Gamma_n. \end{cases}$$

The idea of the proof is to compute Green's function (3.41) associated to $\widetilde{u}$ and interpret the result by means of Lemma 3.9.

Proof. First, we suppose that (E_0, μ_0) is a regular point and $\mu_0 \neq 0$. In this case, we compute

$$(\sigma_+ - \sigma^0)(\sigma_- - \sigma^0) = \frac{\mu^2(F_n^0)^2 - \mu_0^2 F_n^2 - i\mu_0 F_n(F_n^0 F_{n,x} - F_{n,x}^0 F_n) + \frac{(F_n^0 F_{n,x} - F_{n,x}^0 F_n)^2}{4}}{F_n^2 (F_n^0)^2}.$$

We use Corollaries A.1 and A.2 to rewrite the expressions $F_n^0 F_{n,x} - F_{n,x}^0 F_n$ and $\mu^2(F_n^0)^2 - \mu_0^2 F_n^2$. This yields to the equality

$$(\sigma_+ - \sigma^0)(\sigma_- - \sigma^0) = (E - E_0)\frac{\frac{P_{n,x}}{2} + F_n F_n^0 - P_n \sigma^0}{F_n F_n^0}.$$

Finally, we replace this expression in Green's function (3.41):

$$\widetilde{g}(E, \mu, x) = \frac{iF_n(\sigma_+ - \sigma^0)(\sigma_- - \sigma^0)}{2\mu(E - E_0)} = \frac{i\left(F_n + \frac{P_{n,x}}{2F_n^0} - \frac{P_n \sigma^0}{F_n^0}\right)}{2\mu} = \frac{i\widetilde{F}_{\widetilde{n}}}{2\mu}.$$

Since $\widetilde{F}_{\widetilde{n}} = F_n + \frac{P_{n,x}}{2F_n^0} - \frac{P_n\sigma^0}{F_n^0}$ is a polynomial in E of degree n, by means of Lemma 3.9, we conclude that $\widetilde{n} = n$ and $\widetilde{\mu} = \mu$. Thus, $\widetilde{R}_{2\widetilde{n}+1} = R_{2n+1}$.

Now, we suppose that (E_0, μ_0) is a regular point and $\mu_0 = 0$. In this case, we have that $R_{2n+1}^0 = R_{2n+1}(E_0) = 0$ and $\partial_E(R_{2n+1})(E_0) \neq 0$, thus,

$$\mu^2 = R_{2n+1}(E) = (E - E_0)M_{2n},$$

where $M_{2n}(E)$ is a polynomial in E of degree $2n$ such that $M_{2n}(E_0) \neq 0$. Hence, for $\mu_0 = 0$, $\mu^2 = (E - E_0)M_{2n}$ and Corollary A.1, the equality (3.41) becomes

$$\widetilde{g}(E, \mu, x) = \frac{i\left((E - E_0)M_{2n}(F_n^0)^2 + \frac{(E-E_0)^2 P_n^2}{4}\right)}{2\mu(E - E_0)F_n(F_n^0)^2} = \frac{i\left(\frac{M_{2n}}{F_n} + \frac{(E-E_0)P_n^2}{4F_n(F_n^0)^2}\right)}{2\mu}.$$

Now Corollary A.3 guarantees that

$$\frac{M_{2n}}{F_n} + \frac{(E - E_0)P_n^2}{4F_n(F_n^0)^2}$$

is a polynomial in E of degree n. By Lemma 3.9, we obtain that $\widetilde{n} = n$, $\widetilde{\mu} = \mu$ and $\widetilde{R}_{2\widetilde{n}+1} = R_{2n+1}$. Therefore, for regular points we get that $\widetilde{R}_{2\widetilde{n}+1} = R_{2n+1}$ is a polynomial of degree $2n + 1$ in E.

So, by Corollary 3.5 we conclude that $\widetilde{u}$ is solution of a s-KdV$_n$ equation. Thus, a Darboux–Crum transformation with a regular point preserves the spectral curve and the level of the s-KdV hierarchy.

Next, we suppose that (E_0, μ_0) is a singular point of Γ_n, i.e., $\mu_0 = 0$, $R_{2n+1}^0 = R_{2n+1}(E_0) = 0$ and $R_{2n+1,E}^0 = \partial_E(R_{2n+1})(E_0) = 0$, thus,

$$\mu^2 = R_{2n+1}(E) = (E - E_0)^2 Z_{2n-1},$$

where $Z_{2n-1}(E)$ is a polynomial in E of degree $2n-1$. Hence, for $\mu_0 = 0$, $\mu^2 = (E-E_0)^2 Z_{2n-1}$ and Corollary A.1, the equality (3.41) becomes

$$\widetilde{g}(E,\mu,x) = \frac{i\left((E-E_0)^2 Z_{2n-1}(F_n^0)^2 + \frac{(E-E_0)^2 P_n^2}{4}\right)}{2\mu(E-E_0)F_n(F_n^0)^2} = \frac{i\left(\frac{Z_{2n-1}}{F_n} + \frac{P_n^2}{4F_n(F_n^0)^2}\right)}{2(E-E_0)^{-1}\mu}.$$

Now Corollary A.4 guarantees that

$$\frac{Z_{2n-1}}{F_n} + \frac{P_n^2}{4F_n(F_n^0)^2}$$

is a polynomial in E of degree n. By Lemma 3.9, we obtain that $\widetilde{n} = n-1$ and $\widetilde{\mu} = (E-E_0)^{-1}\mu$. Therefore, for singular points we get that $\widetilde{R}_{2\widetilde{n}+1} = (E-E_0)^{-2} R_{2n+1}$ is a polynomial of degree $2n-1$ in E.

Hence, by Corollary 3.5, we conclude that $\widetilde{u}$ is solution of a s-KdV$_{n-1}$ equation. So, a Darboux–Crum transformation with a singular point induces a blowing-up in the spectral curve in this singular point and reduces the level of the s-KdV hierarchy in one. $\square$

Next, we will proceed to establish the situation at the point of infinity $P_\infty = [0:1:0]$ of the spectral curve. For that, we will need to work with the Zariski closure in $\mathbb{P}^2$ of the spectral curve to understand its behaviour under Darboux–Crum transformations for the energy level $E_0 = 0$. In addition, we will use the blowing-up map in $\mathbb{P}^2$ to control the s-KdV level of the transformed potential $\widetilde{u}$.

Let $\pi : \widetilde{\mathbb{P}^2} \to \mathbb{P}^2$ be the blowing-up of $\mathbb{P}^2$ with center $[0:0:1]$. Hence, if $[E:\mu:\nu]$ are homogeneous coordinates in $\mathbb{P}^2$, then the new ones are denoted by $[\widetilde{E}:\widetilde{\mu}:\widetilde{\nu}]$, and π is given by

$$E = \widetilde{E}, \qquad \mu E = \widetilde{\mu}, \qquad \nu = \widetilde{\nu}. \tag{3.46}$$

Theorem 3.15 (II). *Let $P_\infty = [0:1:0]$ be the point of infinity of $\overline{\Gamma}_n$ and u a solution of s-KdV$_n$ equation. Let ϕ_0 be a solution of Schrödinger equation for P_∞ (in particular $E_0 = 0$) and potential u, i.e., $\phi_{0,xx} = u\phi_0$. Let $\widetilde{u} = u - 2(\log\phi_0)_{xx}$ be the Darboux–Crum transformation of u. Then, $\widetilde{u}$ is solution of a s-KdV$_{n+1}$ equation. Futhermore, the spectral curve associated to $\widetilde{u}$ is $\Gamma_{n+1} : \widetilde{\mu}^2 - \widetilde{R}_{2n+3}(E) = 0$, with $\widetilde{R}_{2n+3} = E^2 R_{2n+1}(E)$.*

Proof. First, we consider the homogenized Green's function associated to the transformed Green's function $\widetilde{g}$. Then, by Propositions 3.10 and 3.12, $(\widetilde{g})_h$ is a well defined rational function on $\overline{\Gamma}_n$. But also we have:

$$(\widetilde{g})_h = G_h \circ \pi \qquad \text{on the spectral curve.}$$

Moreover, G_h is a Green's function for the curve defined by $\widetilde{\mu}^2 - \widetilde{R}_{2n+3}(\widetilde{E}) = 0$, where $\widetilde{R}_{2n+3}(\widetilde{E}) = E^2 R_{2n+1}(E)$; that is, for Γ_{n+1}, the strict transform of Γ_n. Observe that $\widetilde{R}_{2n+3} = E^2 R_{2n+1}$ is a polynomial of degree $2n+3$ in E. Then, by Corollary 3.5, we conclude that $\widetilde{u}$ is solution of a s-KdV$_{n+1}$ equation. So, a Darboux–Crum transformation with the point of infinity induces a blowing-down in the spectral curve and increases the level of the s-KdV hierarchy in one. $\square$

Finally we can rewrite 3.14 and 3.15 to establish how the spectral curve $\overline{\Gamma}_n$ behaves under Darboux–Crum transformations.

Theorem 3.16. *Let $P = [E_0 : \mu_0 : \nu_0]$ be a point in $\overline{\Gamma}_n$ and u a solution of s-KdV$_n$ equation. Let ϕ_0 be a solution of Schrödinger equation for E_0 and potential u, say $\phi_{0,xx} = (u - E_0)\phi_0$. Consider $\widetilde{u} = u - 2(\log \phi_0)_{xx}$ the Darboux–Crum transformation of u. Then, $\widetilde{u}$ is solution of a s-KdV$_{\widetilde{n}}$ equation for*

$$\widetilde{n} = \begin{cases} n+1 & \text{if } P = [0:1:0], \\ n & \text{if } P \text{ is a regular point of } \Gamma_n, \\ n-1 & \text{if } P \text{ is an affine singular point of } \Gamma_n. \end{cases}$$

Futhermore, the spectral curve associated to $\widetilde{u}$ is $\Gamma_{\widetilde{n}} : \widetilde{\mu}^2 - \widetilde{R}_{2\widetilde{n}+1} = 0$, with

$$\widetilde{R}_{2\widetilde{n}+1} = \begin{cases} E^2 R_{2n+1} & \text{if } P = [0:1:0], \\ R_{2n+1} & \text{if } P \text{ is a regular point of } \Gamma_n, \\ (E-E_0)^{-2} R_{2n+1} & \text{if } P \text{ is an affine singular point of } \Gamma_n. \end{cases}$$

Example 3.17. Next we apply the previous theorem to a rational s-KdV$_2$ potential. Take the s-KdV$_2$ potential $u = 6x^{-2}$ in the Schrödinger equation (3.20). The spectral curve associated to this potential is $\Gamma_2 : \mu^2 - E^5 = 0$. When $E = 0$, we have the fundamental solutions $\phi_1 = x^{-2}$ and $\phi_2 = x^3$. We consider the Darboux–Crum transformations of u with these solutions:

$$DT(\phi_1)u = u - 2(\log \phi_1)_{xx} = \frac{2}{x^2} = \widetilde{u}_1 \quad \text{and} \quad DT(\phi_2)u = u - 2(\log \phi_2)_{xx} = \frac{12}{x^2} = \widetilde{u}_3.$$

We have that potential $\widetilde{u}_1$ is a solution of s-KdV$_1$ equation. It is well known that the spectral curve associated to this potential is $\Gamma_1 : \mu^2 - E^3 = 0$, the blowing-up of Γ_2 at $(0,0)$. Furthermore, potential $\widetilde{u}_3$ is a solution of s-KdV$_3$ equation, and its associated spectral curve Γ_3 is the blowing-down of Γ_2, that is $\Gamma_3 : \mu^2 - E^7 = 0$.

Now, we take a regular value of E in Γ_2, for instance, $E = -1$. Then, a solution of the Schrödinger equation (3.20) for this value of E is $\phi^+ = \frac{e^x(x^2-3x+3)}{x^2}$. The Darboux–Crum transformation of u with this solution reads:

$$DT(\phi^+)u = u - 2(\log \phi^+)_{xx} = \frac{6(x-1)(x^3-3x^2+3x-3)}{x^2(x^2-3x+3)^2} = \widetilde{u}.$$

Then, this transformed potential is a solution of s-KdV$_2$ equation and the spectral curve associated to this potential is still $\Gamma_2 : \mu^2 - E^5 = 0$.

We sum up this example in the following diagram:

$$\begin{tikzcd}
 & (\widetilde{u}_2, \Gamma_3) & \phi_2 \text{ is a solution for } P_\infty, \\
(u, \Gamma_2) \arrow[ur, "DT(\phi_2)"] \arrow[r, "DT(\phi^+)"] \arrow[dr, "DT(\phi_1)"'] & (\widetilde{u}, \Gamma_2) & \phi^+ \text{ is a solution for a regular point,} \\
 & (\widetilde{u}_1, \Gamma_1) & \phi_1 \text{ is a solution for the affine singular point } (0,0).
\end{tikzcd}$$

Remark 3.18. The importance of Theorem 3.16 lies in the fact that we need to introduce the homogenized Green's function to state it. This new function is the essential tool that allows us to include in our study the point of infinity P_∞ of the affine curve Γ_n. As far as we know, this is a new approach to the understanding of the spectral curve under Darboux transformations.

Similar problems to our result 3.16 were treated by several authors, see [33, Thm 5] and [40, Thm G.2]. In [33], Ehlers and Knörrer studied the action of the Darboux–Crum transformations on the spectral curves by means of the eigenfunctions of the centralizer of the Schrödinger operator.

3.1.4 Galois groups

To finish this section, we briefly discuss how the Darboux–Crum transformations affect the Galois group of the Schrödinger equation

$$(\mathcal{L} - E)\phi = (-\partial_{xx} + u - E)\phi = 0,$$

with potential $u = u(x) \in K$. Let ϕ_1 be a solution of this equation for $E = E_1$. Let L be the Picard–Vessiot field for ϕ_1 and G be the Galois group of this differential fields extension.

Next, take another solution ϕ_0 of Schrödinger equation for $E = E_0$, with $E_0 \neq E_1$ and consider the transformed Schrödinger equation by this solution:

$$(\widetilde{\mathcal{L}} - E)\phi = (-\partial_{xx} + \widetilde{u} - E)\phi = 0,$$

where

$$\widetilde{u} = u - 2(\log \phi_0)_{xx}.$$

Then, the transformed of ϕ_1:

$$\widetilde{\phi}_1 = \phi_{1,x} - \frac{\phi_{0,x}}{\phi_0}\phi_1$$

is a solution of the transformed Schrödinger equation for $E = E_1$. In this case, we have that $\widetilde{u} = \widetilde{u}(x) \in K(\log \phi_0)$ and $\widetilde{\phi}_1 \in L(\log \phi_0)$, i.e, $L(\log \phi_0)$ is the Picard–Vessiot field for $\widetilde{\phi}_1$. Let $\widetilde{G}$ be the Galois group of this differential fields extension. Then, by Lemma 2.3, we have that $\widetilde{G} \subseteq G$.

3.2 Time dependent KdV hierarchy in 1+1 dimensions

Now, we introduce the time dependent KdV hierarchy. In this case, we only present the integrable systems formalism. The differential operators formalism is analogous to that of the stationary case and we refer to [40] Chapter 1.

Consider the derivations ∂_x, $\partial_{t_1}, \partial_{t_2}, \ldots, \partial_{t_m}$ with respect to the variables x, t_1, $t_2, \ldots, t_m$. For each level r of the KdV hierarchy, we shall take the corresponding time t_r and study the corresponding time evolution ∂_{t_r}. Let K_r denote a differential field with compatible derivations ∂_x and ∂_{t_r}, for $r = 1, \ldots, m$, and with field of constants $\mathbf{C}$ algebraically closed and of characteristic zero.

Along this section we take $u = u(x, t_r) \in K_r$ a fixed element of the differential field K_r and $E \in \mathbf{C}$ will be a parameter.

We consider the differential polynomials f_j defined by (3.1) for function $u = u(x, t_r)$ and the Schrödinger equation with this potential

$$(\mathcal{L} - E)\phi = (-\partial_{xx} + u - E)\phi = 0. \tag{3.47}$$

It is well known that the time dependent KdV hierarchy can be constructed by means of the zero curvature condition of the family of integrable systems (see [40] Chapter 1, section 2):

$$\begin{cases} \Phi_x & = U\Phi = \begin{pmatrix} 0 & 1 \\ u - E & 0 \end{pmatrix} \Phi\,, \\ \\ \Phi_{t_r} & = V_r\Phi = \begin{pmatrix} G_r(u) & F_r(u) \\ -H_r(u) & -G_r(u) \end{pmatrix} \Phi\,, \end{cases} \tag{3.48}$$

for F_r, G_r and H_r defined by (3.7), (3.8) and (3.9) respectively.

Just as we did in the stationary case, we fix a level r in the hierarchy and consider the corresponding system (3.48). Its zero curvature condition

$$U_{t_r} - V_{r,x} + [U, V_r] = 0, \tag{3.49}$$

yields to the KdV$_r$ equation

$$\mathrm{KdV}_r : \quad u_{t_r} = -\frac{1}{2}F_{r,xxx} - 2(E-u)F_{r,x} + u_x F_r. \tag{3.50}$$

If we substitute equality (3.7) in this expression, we find:

$$\begin{aligned} u_{t_r} &= -\frac{1}{2}F_{r,xxx} - 2(E-u)F_{r,x} + u_x F_r, \\ &= -\frac{1}{2}\sum_{j=0}^{r} E^j f_{r-j,xxx} - 2(E-u)\sum_{j=0}^{r} E^j f_{r-j,x} + u_x \sum_{j=0}^{r} E^j f_{r-j}, \\ &= \sum_{j=0}^{r} -2E^{j+1} f_{r-j,x} + E^j\left(-\frac{1}{2}f_{r-j,xxx} + 2u f_{r-j,x} + u_x f_{r-j}\right), \\ &= \sum_{j=0}^{r} -2(E^{j+1} f_{r-j,x} - E^j f_{r-j+1,x}) = 2f_{r+1,x}. \end{aligned}$$

Thus,

$$\text{KdV}_r: \quad u_{t_r} = 2f_{r+1,x}. \tag{3.51}$$

Varying $r \in \mathbb{N}$, we get the *time dependent KdV hierarchy*. When u is a solution of KdV_r equation (3.51) for some r, we will say that it is a *KdV potential* or a *KdV$_r$ potential* if it were necessary to specify the level of the hierarchy.

3.2.1 Darboux transformations for the differential polynomials f_j

In this part we present the behaviour of the Darboux–Crum transformations acting on the differential polynomials $f_j(u)$. Here we establish a series of results that will allow us (see Theorems 4.7 and 4.13) to perform Darboux transformations to KdV differential systems in the case we have particular solutions at energy level zero. In this way, we can extend Darboux transformations constructed in Chapter 2 to the only case in which we could not do it.

Let $(\phi_0, \phi_{0,x})^t$ be a column solution of system (3.48) for E_0 and potential $u = u(x, t_r)$. Then, ϕ_0 is a solution of Schrödinger equation (3.47) for E_0. Using this solution we can perform a Darboux–Crum transformation to the potential u. We obtain:

$$DT(\phi_0)u = u - 2(\log\phi_0)_{xx} = u - 2\sigma_{0,x} = \widetilde{u}. \tag{3.52}$$

Now, we consider system (3.48) for this transformed potential:

$$\begin{cases} \Phi_x & = U\Phi = \begin{pmatrix} 0 & 1 \\ \widetilde{u} - E & 0 \end{pmatrix} \Phi\,, \\[2ex] \Phi_{t_r} & = V_r\Phi = \begin{pmatrix} G_r(\widetilde{u}) & F_r(\widetilde{u}) \\ -H_r(\widetilde{u}) & -G_r(\widetilde{u}) \end{pmatrix} \Phi\,. \end{cases} \tag{3.53}$$

The zero curvature condition of this system is still equation (3.51) for $\widetilde{u}$. The differential polynomials $F_r(\widetilde{u})$, $G_r(\widetilde{u})$ and $H_r(\widetilde{u})$ are given by expressions (3.7), (3.8) and (3.9) in terms of $f_j(\widetilde{u})$. We establish the relation between $f_j(\widetilde{u})$ and $f_j(u)$ in the next theorem.

Theorem 3.19. *Let ϕ be a solution of Schrödinger equation* (3.2) *for potential u and fixed energy level E_0. Let $\sigma = (\log\phi)_x$ and $\widetilde{u} = u - 2\sigma_x$ be the Darboux–Crum transformation of u by ϕ. Then*

$$f_j(\widetilde{u}) = f_j(u) + A_j, \qquad \textit{for} \;\; j = 0, 1, 2, \ldots,$$

where A_j is a differential polynomial in u and σ. Moreover, A_j satisfies the differential recursive relations

1. $A_j = -\dfrac{1}{4}A_{j-1,xx} + uA_{j-1} - \dfrac{3}{2}\sigma_x A_{j-1} - \sigma_x f_{j-1}(u),$

2. $A_{j,x} + 2\sigma A_j = -2f_{j,x}(u).$

Proof. We will proceed by induction on j. First, we prove that $f_j(\widetilde{u}) = f_j(u) + A_j$. For $j = 0$ we have $f_0(\widetilde{u}) = 1 = f_0(u)$, then $A_0 = 0$. We suppose it true for j and prove it for $j+1$. Applying equation (3.1) and induction hypothesis we find:

$$\begin{aligned} f_{j+1,x}(\widetilde{u}) &= -\frac{1}{4} f_{j,xxx}(\widetilde{u}) + \widetilde{u} f_{j,x}(\widetilde{u}) + \frac{1}{2}\widetilde{u}_x f_j(\widetilde{u}) \\ &= -\frac{1}{4} f_{j,xxx}(u) + u f_{j,x}(u) + \frac{1}{2} u_x f_j(u) - \frac{1}{4} A_{j,xxx} + u A_{j,x} - 2 f_{j,x}(u)\sigma_x \\ &\quad - 2A_{j,x}\sigma_x + \frac{1}{2} u_x A_j - f_j(u)\sigma_{xx} - A_j \sigma_{xx} = f_{j+1,x}(u) + A_{j+1,x}, \end{aligned}$$

for

$$A_{j+1,x} = -\frac{A_{j,xxx}}{4} + uA_{j,x} - 2f_{j,x}(u)\sigma_x - 2A_{j,x}\sigma_x + \frac{u_x A_j}{2} - f_j(u)\sigma_{xx} - A_j\sigma_{xx}. \quad (3.54)$$

Thus, $f_{j+1}(\widetilde{u}) = f_{j+1}(u) + A_{j+1}$ as we wanted to prove.

Now, we prove statements 1 and 2. We do it by induction and simultaneously. Since $A_0 = 0$ and $f_0(u) = f_0(\widetilde{u}) = 1$, the case $j = 0$ is the trivial one. So, we start the induction process in $j = 1$. For this, we have:

$$f_{1,x}(\widetilde{u}) = -\frac{1}{4} f_{0,xxx}(\widetilde{u}) + \widetilde{u} f_{0,x}(\widetilde{u}) + \frac{1}{2}\widetilde{u}_x f_0(\widetilde{u}) = \frac{1}{2}\widetilde{u}_x.$$

Hence, $f_1(\widetilde{u}) = \frac{\widetilde{u}}{2} + c_1 = \frac{u}{2} - \sigma_x + c_1 = f_1(u) - \sigma_x$, then $A_1 = -\sigma_x$. For $j = 1$ statements 1 and 2 read:

1. $-\dfrac{A_{0,xx}}{4} + uA_0 - \dfrac{3}{2}\sigma_x A_0 - \sigma_x f_0(u) = -\sigma_x = A_1$ and
2. $-2f_{1,x}(u) - A_{1,x} = -u_x + \sigma_{xx} = -2\sigma\sigma_x = 2\sigma A_1$,

by equation (1.20). Now, we suppose both statements true for j and prove them for $j+1$. Derivation with respect to x in the right hand side of statement 1 yields to:

$$\begin{aligned} &-\frac{A_{j,xxx}}{4} + u_x A_j + uA_{j,x} - \frac{3}{2}\sigma_{xx}A_j - \frac{3}{2}\sigma_x A_{j,x} - \sigma_{xx} f_j(u) - \sigma_x f_{j,x}(u) = \\ &\quad = -\frac{A_{j,xxx}}{4} + uA_{j,x} - \sigma_{xx} f_j(u) - \sigma_{xx}A_j - \frac{\sigma_{xx}A_j}{2} + u_x A_j - \frac{3}{2}\sigma_x A_{j,x} - \sigma_x f_{j,x}(u). \end{aligned}$$

Applying equality (1.20) to the term $\sigma_{xx}A_j/2$ we get:

$$\begin{aligned} &-\frac{A_{j,xxx}}{4} + uA_{j,x} - \sigma_{xx} f_j(u) - \sigma_{xx}A_j - \frac{u_x A_j - 2\sigma\sigma_x A_j}{2} + u_x A_j - \frac{3}{2}\sigma_x A_{j,x} - \sigma_x f_{j,x}(u) \\ &\quad = -\frac{A_{j,xxx}}{4} + uA_{j,x} - \sigma_{xx} f_j(u) - \sigma_{xx}A_j + \sigma\sigma_x A_j + \frac{u_x A_j}{2} - \frac{3}{2}\sigma_x A_{j,x} - \sigma_x f_{j,x}(u) \\ &\quad = -\frac{A_{j,xxx}}{4} + uA_{j,x} - \sigma_{xx} f_j(u) - \sigma_{xx}A_j + \frac{u_x A_j}{2} - 2\sigma_x A_{j,x} - \sigma_x f_{j,x}(u) \\ &\quad + \sigma_x(\sigma A_j + \frac{1}{2}A_{j,x}). \end{aligned}$$

Applying induction hypothesis for statement 2 we have:

$$-\frac{A_{j,xxx}}{4}+uA_{j,x}-\sigma_{xx}f_j(u)-\sigma_{xx}A_j+\frac{u_xA_j}{2}-2\sigma_xA_{j,x}-\sigma_xf_{j,x}(u)-\sigma_xf_{j,x}(u)$$
$$=-\frac{A_{j,xxx}}{4}+uA_{j,x}-\sigma_{xx}f_j(u)-\sigma_{xx}A_j+\frac{u_xA_j}{2}-2\sigma_xA_{j,x}-2\sigma_xf_{j,x}(u),$$

which is exactly expression (3.54) for $A_{j+1,x}$. So, we can assume that

$$A_{j+1}=-\frac{A_{j,xx}}{4}+uA_j-\frac{3}{2}\sigma_xA_j-\sigma_xf_j(u).$$

Thus, statement 1 is proved.

Finally, by equations (1.20), (3.1) and (3.54) and induction hypothesis we find for statement 2:

$$\begin{aligned}-2f_{j+1,x}-A_{j+1,x}&=\frac{f_{j,xxx}(u)}{2}-2uf_{j,x}(u)-u_xf_j(u))+\frac{A_{j,xxx}}{4}-uA_{j,x}\\&+2f_{j,x}(u)\sigma_x+2A_{j,x}\sigma_x-\frac{u_xA_j}{2}+f_j(u)\sigma_{xx}+A_j\sigma_{xx}\\&=\left(\frac{f_{j,x}(u)}{2}+\frac{A_{j,x}}{4}\right)_{xx}+(-2f_{j,x}(u)-A_{j,x})(u-\sigma_x)-u_xf_j(u)\\&-\frac{u_xA_j}{2}+A_{j,x}\sigma_x+f_j(u)\sigma_{xx}+A_j\sigma_{xx}\\&=-\frac{\sigma A_{j,xx}}{2}+2u\sigma A_j+A_j\left(\frac{\sigma_{xx}}{2}-\frac{u_x}{2}-2\sigma\sigma_x\right)+f_j(u)(\sigma_{xx}-u_x)\\&=-\frac{\sigma A_{j,xx}}{2}+2u\sigma A_j-3A_j\sigma\sigma_x-2f_j(u)\sigma\sigma_x\\&=2\sigma\left(-\frac{A_{j,xx}}{4}+uA_j-\frac{3}{2}\sigma_xA_j-\sigma_xf_j(u)\right)=2\sigma A_{j+1},\end{aligned}$$

by statement 1. Therefore, statement 2 is proved. This completes the proof. □

Example 3.20. To illustrate the previous theorem we will consider the following KdV$_2$ potentials in the system (3.48). Take $u(x,t_2)=\frac{6(2x^{10}+270x^5t+675t_2^2)}{x^2(x^5-45t_2)^2}$ and a fundamental solution of Schrödinger equation for $E=0$, $\phi_0(x,t_2)=\frac{x^2}{x^5-45t_2}$. Then $\widetilde{u}=6x^{-2}$. Observe that:

$$f_1(u)=\frac{u}{2}=\frac{3(2x^{10}+270x^5t_2+675t_2^2)}{x^2(x^5-45t_2)^2},\quad f_2(u)=-\frac{u_{xx}}{8}+\frac{3}{8}u^2=\frac{45x(x^5+30t_2)}{(x^5-45t_2)^2},$$

and also

$$f_1(\widetilde{u})=\frac{\widetilde{u}}{2}=\frac{3}{x^2},\qquad f_2(\widetilde{u})=-\frac{\widetilde{u}_{xx}}{8}+\frac{3}{8}\widetilde{u}^2=\frac{9}{x^4}.$$

Hence, in this case

$$A_1=f_1(\widetilde{u})-f_1(u)=\frac{-3(x^{10}+360x^5t_2-1350t_2^2)}{x^2(x^5-45t_2)^2},$$
$$A_2=f_2(\widetilde{u})-f_2(u)=\frac{-9(4x^{10}+240x^5t_2-2025t_2^2)}{x^4(x^5-45t_2)^2}.$$

By direct computation we can verify that the functions A_j, $j = 1, 2$, satisfy the retations 3.19 (1) and (2).

Corollary 3.21. *For $i \geq j$ we have the following equality*

$$\sum_{j=0}^{i}(2\sigma A_{i-j} + 2f_{i-j,x}(u) + A_{i-j,x})E^j = 0. \tag{3.55}$$

Theorem 3.19 has several interesting consequences. The main ones are the relations that the transformed potential $\widetilde{u}$ produce for functions $F_r(u)$. Next, we establish some of them, which will be used in the following chapters. In particular, Proposition 3.23 is specially interesting since it gives a relation between σ_x and σ_{t_r}.

Proposition 3.22. *Let A_i and σ be as in Theorem 3.19. For $i = 0, 1, 2, \dots$ we have*

1. *$F_i(\widetilde{u}) = F_i(u) + P_i$, where $P_i = \sum_{j=0}^{i} E^j A_{i-j}$.*
2. *Moreover $P_{i,x} + 2\sigma P_i + 2F_{i,x}(u) = 0$.*

Proof. It is an immediate consequence of Theorem 3.19. □

Proposition 3.23. *Let u be a solution of KdV_r equation and ϕ be a solution of Schrödinger equation* (3.47) *for potential u and energy E_0. Let be $\sigma = (\log \phi)_x$ and $\widetilde{u} = u - 2\sigma_x$ the Darboux–Crum transformation of u. Consider A_{r+1} as defined in 3.19 and P_r as defined in 3.22. Then, we have:*

$$\sigma_{t_r} = -A_{r+1} = \frac{1}{4}P_{r,xx} + EP_r + \sigma_x F_r(u) + \frac{1}{2}P_r(-2u + 3\sigma_x). \tag{3.56}$$

Proof. We prove the first equality. For this, we have $\widetilde{u}_{t_r} = (u - 2\sigma_x)_{t_r} = u_{t_r} - 2\sigma_{x,t_r}$ and $2f_{r+1,x}(\widetilde{u}) = 2f_{r+1,x}(u) + 2A_{r+1,x}$ by Theorem 3.19. Then:

$$2\sigma_{x,t_r} = u_{t_r} - \widetilde{u}_{t_r} = 2f_{r+1,x}(u) - 2f_{r+1,x}(\widetilde{u}) = -2A_{r+1,x},$$

by (3.51). Thus, $\sigma_{t_r} = -A_{r+1}$.

Now, we prove the second equality. Using expression (3.50) for u and $\widetilde{u}$ and applying 3.22 (1), we obtain

$$\begin{aligned}
\widetilde{u}_{t_r} &= -\tfrac{1}{2}F_{r,xxx}(\widetilde{u}) + 2(\widetilde{u} - E)F_{r,x}(\widetilde{u}) + \widetilde{u}_x F_r(\widetilde{u}) \\
&= -\frac{1}{2}F_{r,xxx}(u) + 2(u - E)F_{r,x}(u) + u_x F_r(u) - \frac{1}{2}P_{r,xxx} - 2(E - u)P_{r,x} \\
&- 4\sigma_x F_{r,x}(u) - 4\sigma_x P_{r,x} + u_x P_r - 2\sigma_{xx}F_r(u) - 2\sigma_{xx}P_r \\
&= u_{t_r} - \frac{1}{2}P_{r,xxx} - 2(E - u)P_{r,x} - 4\sigma_x F_{r,x}(u) - 4\sigma_x P_{r,x} + u_x P_r \\
&- 2\sigma_{xx}F_r(u) - 2\sigma_{xx}P_r.
\end{aligned}$$

Since $2\sigma_{x,t_r} = u_{t_r} - \widetilde{u}_{t_r}$, we have

$$2\sigma_{x,t_r} = \frac{1}{2}P_{r,xxx} + 2EP_{r,x} - 2uP_{r,x} + 4\sigma_x F_{r,x}(u) + 4\sigma_x P_{r,x} - u_x P_r + 2\sigma_{xx}F_r(u) + 2\sigma_{xx}P_r.$$

Applying 3.22 (2) to the expresion $\sigma_x P_{r,x}$, we find:

$$\begin{aligned}2\sigma_{x,t_r} &= \frac{1}{2}P_{r,xxx} + 2EP_{r,x} - 2uP_{r,x} + 4\sigma_x F_{r,x}(u) + 3\sigma_x P_{r,x}\\ &+ \sigma_x(-2\sigma P_r - 2F_{r,x}(u)) - u_x P_r + 2\sigma_{xx} F_r(u) + 2\sigma_{xx} P_r\\ &= \frac{1}{2}P_{r,xxx} + 2EP_{r,x} + 2(\sigma_{xx}F_r(u) + \sigma_x F_{r,x}(u)) + P_{r,x}(-2u + 3\sigma_x)\\ &+ P_r(-2\sigma\sigma_x - u_x + 2\sigma_{xx}).\end{aligned}$$

Moreover, for the coefficient of P_r of this expression we have:

$$-2\sigma\sigma_x - u_x + 2\sigma_{xx} = (-\sigma^2 - u + 2\sigma_x)_x = (-2u + 3\sigma_x)_x$$

by (1.20). Thus, we obtain

$$2\sigma_{x,t_r} = \left(\frac{1}{2}P_{r,xx} + 2EP_r + 2\sigma_x F_r(u) + P_r(-2u + 3\sigma_x)\right)_x .$$

Hence we have proved the statement. □

We finish this part with the following result. It makes a connection between differential polynomials $f_r(u)$ and some differential polynomials $g_r(\sigma)$ defined by

$$g_r(\sigma) := -A_{r+1} = \frac{1}{2}P_{r,xx} + 2EP_r + 2\sigma_x F_r(u) + P_r(-2u + 3\sigma_x). \tag{3.57}$$

Proposition 3.24. *We have the following relations:*

1. $(2\sigma + \partial_x)g_r(\sigma) = 2f_{r+1,x}(u) = -\frac{1}{2}F_{r,xxx}(u) + 2(u - E)F_{r,x}(u) + u_x F_r(u)$,
2. $(2\sigma - \partial_x)g_r(\sigma) = 2f_{r+1,x}(\widetilde{u}) = -\frac{1}{2}F_{r,xxx}(\widetilde{u}) + 2(\widetilde{u} - E)F_{r,x}(\widetilde{u}) + \widetilde{u}_x F_r(\widetilde{u})$.

Proof. Statement (1) is just statement (2) of Theorem 3.19 rewritten.

For statement (2) we have:

$$\begin{aligned}2f_{r+1,x}(\widetilde{u}) &= 2f_{r+1,x}(u) + 2A_{r+1,x} = 2\sigma g_r(\sigma) + g_{r,x}(\sigma) - 2g_{r,x}(\sigma)\\ &= 2\sigma g_r(\sigma) - g_{r,x}(\sigma) = (2\sigma - \partial_x)g_r(\sigma)\end{aligned}$$

by statement (1) and equation (3.57). □

3.2.2 Galois groups

To finish this section we make a brief analysis about the connection between the Galois groups of system (3.48) and its transformed system (3.53).

As we have shown in Section 2.4, both systems can be transformed into AKNS systems by applying the gauge transformation

$$R = \begin{pmatrix} \lambda & 1 \\ -1 & 0 \end{pmatrix}.$$

Let $(\mathfrak{s})$ be the AKNS system associated to (3.48) and $(\mathfrak{s}_1)$ the AKNS system associated to (3.53). Now, consider the matrix Darboux transformation $D = J(\lambda I - \Sigma)$ for $(\mathfrak{s})$, as in (2.26), for a generic value λ_0 such that $\lambda_0^2 + E_0 = 0$, for ϕ_0 and E_0 the same as in (3.52). We denote the transformed system of $(\mathfrak{s})$ by D as $(\widetilde{\mathfrak{s}})$. Then, in Section 2.5 we proved that $(\widetilde{\mathfrak{s}}) = (\mathfrak{s}_1)$.

Let G be the Galois group of system (3.48) and $\widetilde{G}$ the Galois group of system (3.53). Then, by Theorem 2.2, we get that $\widetilde{G} \subseteq G$.

Chapter 4

Schrödinger operators with KdV rational potentials

Consider the derivations ∂_x, $\partial_{t_1}, \partial_{t_2}, \ldots, \partial_{t_m}$ with respect to the variables x and $t_1, \ldots, t_m$. Along this chapter K_r will denote the differential field of rational functions in the variables x and t_r with compatible derivations ∂_x and ∂_{t_r}, for $r = 1, \ldots, m$. Let us assume that its field of constants $\mathbf{C}$ is algebraically closed and of characteristic zero. Let $E \in \mathbf{C}$ be a parameter.

In this chapter we give a fundamental matrix for the system (3.48):

$$\begin{cases} \Phi_x &= U\Phi = \begin{pmatrix} 0 & 1 \\ u - E & 0 \end{pmatrix} \Phi, \\ \Phi_{t_r} &= V_r\Phi = \begin{pmatrix} -\dfrac{F_{r,x}(u)}{2} & F_r(u) \\ (u-E)F_r(u) - \dfrac{F_{r,xx}(u)}{2} & \dfrac{F_{r,x}(u)}{2} \end{pmatrix} \Phi, \end{cases} \tag{4.1}$$

with Adler–Moser potentials (as defined in [7]) depending on the energy level E. We will see that for every $E \neq 0$ the fundamental matrix associated to the system presents a similar algebraic structure. A fundamental matrix for $E = 0$ can be also computed. However, it will not be obtained by a specialization process from the fundamental matrix obtained for $E \neq 0$. The spectral curve is the tool that will allow us to understand why fundamental matrices present these different behaviours according to the values of the energy. As far as we know these computations are new results. We have used SAGE to obtain the formulas in the examples.

Finally, we will compute the Galois groups for the differential extensions given by the fundamental matrices for each value of E.

The work exposed here is based on a joint work with Sonia Jiménez, Juan J. Morales and María Ángeles Zurro. The results that appear in this chapter are contained in the preprint [53] and in the article [52].

4.1 Adler–Moser rational potentials

In this section we review the KdV$_r$ rational potentials that Adler and Moser construct in [7]. These are potentials u_n for the Schrödinger equation

$$(\mathcal{L} - E)\phi = (-\partial_{xx} + u - E)\phi = 0 \tag{4.2}$$

of the form

$$u_n = -2(\log \theta_n)_{xx}, \tag{4.3}$$

where θ_n are functions in x defined by the differential recursion:

$$\theta_0 = 1, \qquad \theta_1 = x, \qquad \theta_{n+1,x}\theta_{n-1} - \theta_{n+1}\theta_{n-1,x} = (2n+1)\theta_n^2. \tag{4.4}$$

Each time we iterate this recursion, we will obtain an integration constant of x, which we denote by τ_i. So the functions θ_n and the potentials u_n are in fact functions in $x, \tau_2, \ldots, \tau_n$. As we will show next, the integration constants $\tau_2, \ldots, \tau_n$ depend on t_r.

Next result proves that this recursion always has a solution. Moreover, the solutions are polynomials in $x, \tau_2, \ldots, \tau_n$ with coefficients in the field $\mathbf{C}$.

Lemma 4.1. *Let be $F = \mathbf{C}[x, \tau_2, \ldots, \tau_k]$, $a \in \mathbf{C}^*$ and $b \in \mathbf{C}$. Let (F, ∂_x) be the ring of polynomials in the variables $x, \tau_2, \ldots, \tau_k$ with derivative ∂_x, whose field of constants is $\mathbf{C}$. Let consider the sequence defined recursively by:*

$$P_0 = 1, \qquad P_1 = ax + b, \qquad P_{n+1,x}P_{n-1} - P_{n+1}P_{n-1,x} = (2n+1)P_n^2. \tag{4.5}$$

Then $P_n \in F$ for all n.

The proof of this lemma is an easy extension of the proof of Lemma 2 in [7].

Applying this lemma for $a = 1$ and $b = 0$, we obtain that functions θ_n are polynomials in $x, \tau_2, \ldots, \tau_n$ with coefficients in $\mathbf{C}$ for all n. The first terms of the recursion are

n	θ_n
0	1
1	x
2	$x^3 + \tau_2$
3	$x^6 + 5\tau_2 x^3 + \tau_3 x - 5\tau_2^2.$

Hereafter we will refer to polynomials θ_n as *Adler–Moser polynomials* and to functions

$$u_n := -2(\log \theta_n)_{xx}, \tag{4.6}$$

defined by means of lemma 4.1, as *KdV rational solitons* or *Adler–Moser potentials*.

Adler and Moser proved in [7] that for each r there exist a choice of τ_j, $j = 2, \ldots, n$, such that every potential u_n is a solution of the KdV$_r$ equation (3.51) for constants $c_i = 0$, $i = 1, \ldots, r$. Their theorem reads as follows, where the function g_r is defined by (3.57):

Theorem 4.2 (Theorem 2, [7]). *There is a unique choice of rational functions $\gamma_{rj}(\tau_2, \dots, \tau_j)$ and differential operators*

$$\Xi_r = \sum_{j=1}^{\infty} \gamma_{rj} \frac{\partial}{\partial \tau_j}$$

such that $2f_{r+1,x}(u_n) = \Xi_r u_n$ *for* $n = 0, 1, 2, \dots$, *and*

$$g_r(v_n) = \Xi_r(v_n) \quad \text{where} \quad v_n = \frac{\theta_{n+1,x}}{\theta_{n+1}} - \frac{\theta_{n,x}}{\theta_n}. \tag{4.7}$$

(Since u_n and v_n depend only on finitely many variables the sum breaks off.) In other words, if the τ_j satisfy

$$\frac{d\tau_j}{dt_r} = \gamma_{rj}(\tau_2, \dots, \tau_j), \quad j \le n,$$

then $u_n = u_n(\tau_2, \dots, \tau_n)$ *solves the equation* $u_{t_r} = 2f_{r+1,x}(u)$.

Remark 4.3. Theorem 4.2 shows that for each level r, the formula (4.6) for θ_n is a solution of the KdV_r equation. Hence the constants $\tau_2, \dots, \tau_j \in \mathbf{C}(t_r)$ must be adapted in terms of t_r to get a solution of the KdV_r equation. When this is the case, functions $\theta_n(x, t_r) \in K_r$ will be polynomials in x and rational functions in t_r and potentials $u_n(x, t_r) \in K_r$ will be rational functions in x and t_r.

We will call *adjusted polynomials* $\theta_{r,n} = \theta_{r,n}(x, t_r)$ and *adjusted potentials* $u_{r,n} = u_{r,n}(x, t_r) = -2(\log \theta_{r,n})_{xx}$, whenever the functions $\tau_2, \dots, \tau_j \in \mathbf{C}(t_r)$ are fixed by means of Theorem 4.2. Otherwise, we will call *unadjusted polynomials* $\theta_n = \theta_n(x, t_r)$ and *unadjusted potentials* $u_n = u_n(x, t_r) = -2(\log \theta_n)_{xx}$.

Example 4.4. As an example of adjusted potentials, we show the first Adler–Moser potentials for $r = 1$ with the explicit choice of functions $\tau_2, \dots, \tau_n$. These potentials are solutions of the KdV_1 equation for $c_1 = 0$: $u_{t_1} = \frac{3}{2} u u_x - \frac{1}{4} u_{xxx}$.. The computations were made using SAGE. We have

n	u_n	$(\tau_2, \cdots, \tau_n)$
0	0	
1	$\dfrac{2}{x^2}$	
2	$\dfrac{6x(x^3 - 6t_1)}{(x^3 + 3t_1)^2}$	$(3t_1)$
3	$\dfrac{6x(2x^9 + 675x^3 t_1^2 + 1350 t_1^3)}{(x^6 + 15x^3 t_1 - 45 t_1^2)^2}$	$(3t_1, 0)$
4	$\dfrac{10 p_4(x, t_1)}{(x^{10} + 45x^7 t_1 + 4725 x t_1^3)^2}$	$(3t_1, 0, 0)$
5	$\dfrac{30 x p_5(x, t_1)}{(\theta_5)^2}$	$(3t_1, 0, 0, 33075 t_1^3)$

where

$$
\begin{aligned}
p_4(x,t_1) &= 2x^{18} + 72x^{15}t_1 + 2835x^{12}t_1^2 - 66150x^9t_1^3 - 1190700x^6t_1^4 + 4465125t_1^6,\\
p_5(x,t_1) &= x^{27} + 126x^{24}t_1 + 7560x^{21}t_1^2 + 5655825x^{15}t_1^4 + 500094000x^{12}t_1^5,\\
&\quad + 4313310750x^9t_1^6 + 11252115000x^6t_1^7 + 295368018750x^3t_1^8,\\
&\quad - 590736037500t_1^9,\\
\theta_5 &= x^{15} + 105x^{12}t_1 + 1575x^9t_1^2 + 33075x^6t_1^3 - 992250x^3t_1^4 - 1488375t_1^5.
\end{aligned}
$$

We notice that the adjustment of τ_i is not linear in t_1.

4.1.1 Spectral problems associated to the time dependent KdV hierarchy for Adler–Moser potentials

In subsection 3.1.1 we have shown how to associate to system (3.6) a curve, the so-called spectral curve. We can also associate a curve to system (4.1) for potentials $u_{r,n}$ (recall that this potentials are solutions of KdV$_r$ equation). For this, we consider the stationary problem associated to system (4.1) for $t_r = 0$. We denote the corresponding stationary potential as $u_{r,n}^{(0)}(x) = u_{r,n}(x, t_r = 0)$.

We have the following result for Adler-Moser potentials $u_{r,n}$ in the stationary case:

Lemma 4.5. *For $\tau_j = 0$, $j = 2,\dots,n$, we have*

$$\theta_n(x,0) = \theta_n^{(0)}(x) = x^{n(n+1)/2} \quad \text{and} \quad u_{r,n}^{(0)}(x) = u_{r,n}(x, t_r = 0) = \frac{n(n+1)}{x^2}. \tag{4.8}$$

Proof. We first prove that, when $\tau_j = 0$, $j = 2,\dots,n$, we have that $\theta_n(x,0) = \theta_n^{(0)}(x) = x^{n(n+1)/2}$.

We prove it by induction on n. For $n = 0$ we have $\theta_0^{(0)} = 1 = x^{0\cdot 1/2}$. For $n = 1$ we have that $\theta_1^{(0)} = x = x^{1\cdot 2/2}$.

We suppose it true for $n-1$ and n and prove it for $n+1$. For this, we use the recursion formula (4.4):

$$\theta_{n+1,x}^{(0)} x^{(n-1)n/2} - \theta_{n+1}^{(0)} \frac{(n-1)n}{2} x^{(n-1)n/2-1} = (2n+1)x^{n(n+1)}.$$

This expression simplifies to

$$\theta_{n+1,x}^{(0)} x - \theta_{n+1}^{(0)} \frac{(n-1)n}{2} = (2n+1)x^{(n+1)(n+2)/2}.$$

The solution of this differential equation is $\theta_{n+1}^{(0)} = x^{(n+1)(n+2)/2}$.

Now, we compute the potentials $u_n^{(0)}$:

$$u_n^{(0)} = -2(\log \theta_n^{(0)})_{xx} = -2(\log x^{n(n+1)/2})_{xx} = \frac{n(n+1)}{x^2}.$$

As we wanted to prove. □

The first level of the stationary KdV hierarchy for which potentials $u_{r,n}^{(0)}(x) = n(n+1)x^{-2}$ defined in the aforementioned Lemma are solutions of is level n, which implies that in the stationary case we will have $r = n$. Hence, we will denote them just by $u_n^{(0)}(x)$.

In this way, we have seen that the stationary part of system (4.1) for potential $u_{r,n}$ is system (3.6) for potential $u_n^{(0)}(x)$. Therefore, paired with the KdV$_r$ equation problem we have a s-KdV$_n$ equation problem.

The corresponding spectral curve to this s-KdV$_n$ equation problem is

$$\Gamma_n : \mu^2 - E^{2n+1} = 0. \tag{4.9}$$

So, we take this spectral curve to be the curve associated to the time dependent problem for Adler-Moser potentials. Observe that $(E, \mu) = (0, 0)$ is the unique affine singular point of Γ_n.

Example 4.6. Potential $u_2 = -2(\log\theta_2)_{xx} = \frac{6x(x^3-2\tau_2)}{(x^3+\tau_2)^2}$ is a solution of KdV$_1$ equation for $\tau_2 = 3t_1$:

$$u_{2,t_1} = \frac{3}{2}u_2 u_{2,x} - \frac{1}{4}u_{2,xxx}.$$

For $\tau_2 = 0$ we have that $u_2^{(0)} = 6x^{-2}$ is a solution of s-KdV$_2$ equation and the corresponding spectral curve is

$$\Gamma_2 : \mu^2 - E^5 = 0.$$

Hence, the curve associated to the temporary problem for $u = u_2$ is Γ_2.

4.2 Fundamental matrices for Schrödinger equation with KdV$_r$ potentials. Case $E = 0$

In this section, we explicitly compute fundamental matrices for system (4.1) for $r \geq 1$ when the potential u is $u_{r,n} = -2(\log\theta_{r,n})_{xx}$ and $E = 0$, and prove some relevant properties of them. Recall that $u_{r,n} = u_{r,n}(x, t_r)$ is a solution of KdV$_r$ equation (see Remark 4.3). Hence, we study the system

$$\begin{cases} \Phi_x &= U\Phi = \begin{pmatrix} 0 & 1 \\ u_{r,n} & 0 \end{pmatrix}\Phi, \\ \Phi_{t_r} &= V_r\Phi = \begin{pmatrix} -\dfrac{f_{r,x}(u_{r,n})}{2} & f_r(u_{r,n}) \\ u_{r,n}f_r(u_{r,n}) - \dfrac{f_{r,xx}(u_{r,n})}{2} & \dfrac{f_{r,x}(u_{r,n})}{2} \end{pmatrix}\Phi. \end{cases} \tag{4.10}$$

It is obvious that the zero curvature condition of this system is the KdV$_r$ equation for $c_i = 0$, $i = 1, \ldots, r$:

$$\partial_{t_r} u_{r,n} = 2f_{r+1,x}(u_{r,n}). \tag{4.11}$$

From now on we will denote $u_{r,n,t_r} = \partial_{t_r}(u_{r,n})$. We have the following result:

Theorem 4.7. *Let n be a non negative integer. For $E = 0$ and $u = u_{r,n}$ a fundamental matrix for system* (4.10) *is:*

$$\mathcal{B}_{n,0}^{(r)} = \begin{pmatrix} \phi_{1,r,n} & \phi_{2,r,n} \\ \phi_{1,r,n,x} & \phi_{2,r,n,x} \end{pmatrix}, \tag{4.12}$$

where

$$\phi_{1,r,n}(x,t_r) = \frac{\theta_{r,n-1}}{\theta_{r,n}} \qquad \text{and} \qquad \phi_{2,r,n}(x,t_r) = \frac{\theta_{r,n+1}}{\theta_{r,n}}. \tag{4.13}$$

For $n = 0$ we define $\theta_{r,-1} := 1$. We notice that $\phi_{2,r,n} = (\phi_{1,r,n+1})^{-1}$.

Proof. We prove it by induction on n. For $n = 0$ the definition $\theta_{r,0} = 1$ gives $u_{r,0} = 0$. So, system (4.10) reads

$$\begin{cases} \begin{pmatrix} \phi_{1,r,0,x} & \phi_{2,r,0,x} \\ \phi_{1,r,0,xx} & \phi_{2,r,0,xx} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \phi_{1,r,0} & \phi_{2,r,0} \\ \phi_{1,r,0,x} & \phi_{2,r,0,x} \end{pmatrix} = \begin{pmatrix} \phi_{1,r,0,x} & \phi_{2,r,0,x} \\ 0 & 0 \end{pmatrix}, \\ \begin{pmatrix} \phi_{1,r,0,t_r} & \phi_{2,r,0,t_r} \\ \phi_{1,r,0,xt_r} & \phi_{2,r0,xt_r} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \phi_{1,r,0} & \phi_{2,r,0} \\ \phi_{1,r,0,x} & \phi_{2,r,0,x} \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \end{cases}$$

Thus, $\phi_{1,r,0} = 1$ and $\phi_{2,r,0} = x$ generate $\mathcal{B}_{0,0}^{(r)}$. Since $\theta_{r,1} = x$ we have that $\phi_{1,r,0} = \frac{\theta_{r,-1}}{\theta_{r,0}}$ and $\phi_{2,r,0} = \frac{\theta_{r,1}}{\theta_{r,0}}$.

Now, we suppose it true for n and prove it for $n+1$. For n we know that $\phi_{1,r,n} = \frac{\theta_{r,n-1}}{\theta_{r,n}}$ and $\phi_{2,r,n} = \frac{\theta_{r,n+1}}{\theta_{r,n}}$ generate $\mathcal{B}_{n,0}^{(r)}$. Therefore, $\phi_{1,r,n}$ and $\phi_{2,r,n}$ are solutions of the Schrödinger equation $\phi_{xx} = u_{r,n}\phi$. We apply a Darboux–Crum transformation with $\phi_{2,r,n}$ to this Schrödinger equation and we obtain:

$$\begin{aligned} DT(\phi_{2,r,n})u_{r,n} &= u_{r,n} - 2(\log \phi_{2,r,n})_{xx} = -2(\log \theta_{r,n})_{xx} - 2(\log \phi_{2,r,n})_{xx} \qquad (4.14) \\ &= -2(\log \phi_{2,r,n}\theta_{r,n})_{xx} = -2(\log \theta_{r,n+1})_{xx} = u_{r,n+1}, \\ DT(\phi_{2,r,n})\phi_{1,r,n} &= \phi_{1,r,n,x} - \frac{\phi_{2,r,n,x}}{\phi_{2,r,n}}\phi_{1,r,n} = -(2n+1)\frac{\theta_n}{\theta_{r,n+1}} \qquad (4.15) \\ &= -(2n+1)\phi_{1,r,n+1}. \end{aligned}$$

So, $\phi_{1,r,n+1} := \frac{\theta_{r,n}}{\theta_{r,n+1}}$ is a solution of $\phi_{xx} = u_{r,n+1}\phi$, and, obviously, the matrix column $(\phi_{1,r,n+1},\ \phi_{1,r,n+1,x})^t$ is a column solution of the first equation of the system for $u_{r,n+1}$. Now we verify that this column matrix is also a solution of the second equation:

$$\begin{aligned} \begin{pmatrix} \phi_{1,r,n+1,t_r} \\ \phi_{1,r,n+1,x,t_r} \end{pmatrix} &= \begin{pmatrix} -\frac{f_{r,x}(u_{r,n+1})}{2} & f_r(u_{r,n+1}) \\ u_{r,n+1}f_r(u_{r,n+1}) - \frac{f_{r,xx}(u_{r,n+1})}{2} & \frac{f_{r,x}(u_{r,n+1})}{2} \end{pmatrix} \cdot \begin{pmatrix} \phi_{1,r,n+1} \\ \phi_{1,r,n+1,x} \end{pmatrix} \\ &= \begin{pmatrix} -\frac{f_{r,x}(u_{r,n+1})}{2}\phi_{1,r,n+1} + f_r(u_{r,n+1})\phi_{1,r,n+1,x} \\ \left(u_{r,n+1}f_r(u_{r,n+1}) - \frac{f_{r,xx}(u_{r,n+1})}{2}\right)\phi_{1,r,n+1} + \frac{f_{r,x}(u_{r,n+1})}{2}\phi_{1,r,n+1,x} \end{pmatrix}. \end{aligned}$$

We notice that the second row is just the partial derivative with respect to x of the first one. Hence, we just have to check that expressions (4.14) and (4.15) satisfy equation

$$(\phi_{1,r,n+1})_{t_r} = -\frac{f_{r,x}(u_{r,n+1})}{2}\phi_{1,r,n+1} + f_r(u_{r,n+1})\phi_{1,r,n+1,x}. \tag{4.16}$$

Using expression (4.15), on the left hand side of this equation we have:

$$\begin{aligned}\phi_{1,r,n+1,t_r} &= \frac{1}{2n+1}\left(\frac{\phi_{2,r,n,x}}{\phi_{2,r,n}}\phi_{1,r,n} - \phi_{1,r,n,x}\right)_{t_r}\\ &= \frac{1}{2n+1}\left(\phi_{1,r,n,t_r}\frac{\phi_{2,r,n,x}}{\phi_{2,r,n}} + \phi_{1,r,n}\left(\frac{\phi_{2,r,n,x,t_r}}{\phi_{2,r,n}} - \frac{\phi_{2,r,n,x}\phi_{2,r,n,t_r}}{\phi_{2,r,n}^2}\right) - \phi_{1,r,n,x,t_r}\right),\end{aligned}$$

then, applying induction hypothesis we get

$$\phi_{1,r,n+1,t_r} = \frac{1}{2n+1}\left(\phi_{1,r,n}\frac{\phi_{2,r,n,x}}{\phi_{2,r,n}} - \phi_{1,r,n,x}\right)\cdot\left(\frac{f_{r,x}(u_{r,n})}{2} - f_r(u_{r,n})\frac{\phi_{2,r,n,x}}{\phi_{2,r,n}}\right). \tag{4.17}$$

Analogously, applying expression (4.15) and induction hypothesis, on the right hand side we obtain:

$$\begin{aligned}&-\frac{f_{r,x}(u_{r,n+1})}{2}\phi_{1,r,n+1} + f_r(u_{r,n+1})\phi_{1,r,n+1,x} =\\ &= \frac{1}{2n+1}\left(\phi_{1,r,n}\frac{\phi_{2,r,n,x}}{\phi_{2,r,n}} - \phi_{1,r,n,x}\right)\cdot\left(-\frac{f_{r,x}(u_{r,n+1})}{2} - f_r(u_{r,n+1})\frac{\phi_{2,r,n,x}}{\phi_{2,r,n}}\right).\end{aligned} \tag{4.18}$$

Now, we prove that expressions (4.17) and (4.18) are equal. Applying Theorem 3.19 statement 2 for $\sigma = \frac{\phi_{2,r,n,x}}{\phi_{2,r,n}}$ to expression (4.18) leads to:

$$\begin{aligned}-\frac{f_{r,x}(u_{r,n+1})}{2} - f_r(u_{r,n+1})\frac{\phi_{2,r,n,x}}{\phi_{2,r,n}} &= -\frac{f_{r,x}(u_{r,n}) + A_{r,x}}{2} - (f_r(u_{r,n}) + A_r)\frac{\phi_{2,r,n,x}}{\phi_{2,r,n}}\\ &= -\frac{f_{r,x}(u_{r,n})}{2} - f_r(u_{r,n})\frac{\phi_{2,r,n,x}}{\phi_{2,r,n}} - \frac{A_{r,x}}{2} - A_r\frac{\phi_{2,r,n,x}}{\phi_{2,r,n}}\\ &= -\frac{f_{r,x}(u_{r,n})}{2} - f_r(u_{r,n})\frac{\phi_{2,r,n,x}}{\phi_{2,r,n}} - \frac{A_{r,x}}{2} + f_{r,x}(u_{r,n}) + \frac{A_{r,x}}{2}\\ &= \frac{f_{r,x}(u_{r,n})}{2} - f_r(u_{r,n})\frac{\phi_{2,r,n,x}}{\phi_{2,r,n}},\end{aligned}$$

which is equal to expression (4.17). Therefore, both sides of expression (4.16) coincide.

Now we proceed as in [7]. We take another column solution $(\phi_{2,r,n+1},\ \phi_{2,r,n+1,x})^t$ of this system for potential $u_{r,n+1}$ which is linearly independent of the one we have just computed, i.e., $\det \mathcal{B}^{(r)}_{n+1,0}$ is a nontrivial constant. We take $\phi_{2,r,n+1}$ such that

$$\det \mathcal{B}^{(r)}_{n+1,0} = 2(n+1) + 1.$$

We notice that, with this condition we have:

$$\det \mathcal{B}^{(r)}_{n+1,0} = \phi_{2,r,n+1,x}\frac{\theta_{r,n}}{\theta_{r,n+1}} - \phi_{2,r,n+1}\frac{\theta_{r,n,x}\theta_{r,n+1} - \theta_{r,n}\theta_{r,n+1,x}}{\theta^2_{r,n+1}} = 2(n+1)+1,$$

multiplying both sides by $\theta^2_{r,n+1}$ and using the recursion formula (4.4) we get:

$$\phi_{2,r,n+1,x}\theta_{r,n}\theta_{r,n+1} - \phi_{2,r,n+1}(\theta_{r,n,x}\theta_{r,n+1} - \theta_{r,n}\theta_{r,n+1,x}) = \theta_{r,n+2,x}\theta_{r,n} - \theta_{r,n+2}\theta_{r,n,x}.$$

Setting $\phi_{2,r,n+1} = \dfrac{\alpha_{2,r,n+1}}{\theta_{r,n+1}}$ yields to:

$$\alpha_{2,r,n+1,x}\theta_{r,n} - \alpha_{2,r,n+1}\theta_{r,n,x} = \theta_{r,n+2,x}\theta_{r,n} - \theta_{r,n+2}\theta_{r,n,x},$$

thus, $\alpha_{2,r,n+1} = \theta_{r,n+2}$ and $\phi_{2,r,n+1} = \frac{\theta_{r,n+2}}{\theta_{r,n+1}}$. This concludes the proof. □

Remark 4.8. Notice that, since we are taking particular solutions of the Schrödinger equation for $E = 0$ to perform the Darboux–Crum transformations, we can not construct the Darboux matrix as we showed in Chapter 2 in order to transform the complete system at once. That is why we have to perform a Darboux–Crum transformation, in the sense of Section 1.3, with this solution and to prove that the transformed functions are solutions of the second equation of the system as well. For that, it is necessary to control the action of the Darboux–Crum transformations over the differential polynomials f_j, as we showed in Section 3.2.1.

Adler and Moser proved in [7] that matrix $\mathcal{B}^{(r)}_{n,0}$ is a fundamental matrix for the Schrödinger equation for $E = 0$. But they did not prove there that this matrix is also a fundamental matrix for the second equation of the system (4.10).

Remark 4.9. Since $\phi_{1,r,n} = \frac{\theta_{r,n-1}}{\theta_{r,n}}$ and $\phi_{2,r,n} = \frac{\theta_{r,n+1}}{\theta_{r,n}}$ are solutions of Schrödinger equation (4.2) for $E = 0$, this translates into the following equation for polynomials $\theta_{r,n}$:

$$\theta_{r,n+1,xx}\theta_{r,n} + \theta_{r,n+1}\theta_{r,n,xx} - 2\theta_{r,n,x}\theta_{r,n+1,x} = 0. \tag{4.19}$$

We obtain the following expression for the determinant of fundamental matrices $\mathcal{B}^{(r)}_{n,0}$:

Theorem 4.10. *We have that*

$$\det \mathcal{B}^{(r)}_{n,0} = 2n+1. \tag{4.20}$$

Example 4.11. To illustrate this case, we present explicit computations using SAGE of fundamental solutions of the system for $n = 0, 1, 2$ and 3.

1. First, we show the first examples of fundamental solutions for unadjusted functions θ_n:

$$
\begin{array}{ccc}
\phi_{1,r,n} & \phi_{2,r,n} & u_{r,n} \\
1 & x & 0 \\
\dfrac{1}{x} & \dfrac{x^3+\tau_2}{x} & \dfrac{2}{x^2} \\
\dfrac{x}{x^3+\tau_2} & \dfrac{x^6+5x^3\tau_2+x\tau_3-5\tau_2^2}{x^3+\tau_2} & \dfrac{6x(x^3-2\tau_2)}{(x^3+\tau_2)^2} \\
\dfrac{x^3+\tau_2}{x^6+5x^3\tau_2+x\tau_3-5\tau_2^2} & \dfrac{p_1(x,\tau_2,\tau_3,\tau_4)}{x^6+5x^3\tau_2+x\tau_3-5\tau_2^2} & \dfrac{p_2(x,\tau_2,\tau_3)}{(x^6+5x^3\tau_2+x\tau_3-5\tau_2^2)^2}
\end{array}
$$

where

$$
\begin{aligned}
p_1(x,\tau_2,\tau_3,\tau_4) &= x^{10}+15x^7\tau_2+7x^5\tau_3-35x^2\tau_2\tau_3+175x\tau_2^3-\frac{7}{3}\tau_3^2+x^3\tau_4+\tau_2\tau_4, \\
p_2(x,\tau_2,\tau_3) &= 12x^{10}-36x^5\tau_3+450x^4\tau_2^2+300x\tau_2^3+2\tau_3^2.
\end{aligned}
$$

2. Next, we compute fundamental solutions for potentials which are solutions of the first level of the KdV hierarchy, KdV$_1$ equation: $u_{t_1} = \frac{3}{2}uu_x - \frac{1}{4}u_{xxx}$. We also show the explicit choice of the functions τ_i.

$$
\begin{array}{cccc}
\phi_{1,1,n} & \phi_{2,1,n} & u_{1,n} & (\tau_2,\ldots,\tau_n) \\
1 & x & 0 & \\
\dfrac{1}{x} & \dfrac{x^3+3t_1}{x} & \dfrac{2}{x^2} & (3t_1) \\
\dfrac{x}{x^3+3t_1} & \dfrac{x^6+15x^3t_1-45t_1^2}{x^3+3t_1} & \dfrac{6x(x^3-6t_1)}{(x^3+3t_1)^2} & (3t_1,0) \\
\dfrac{x^3+3t_1}{x^6+15x^3t_1-45t_1^2} & \dfrac{x^{10}+45x^7t_1+4725xt_1^3}{x^6+15x^3t_1-45t_1^2} & \dfrac{6x(2x^9+675x^3t_1^2+1350t_1^3)}{(x^6+15x^3t_1-45t_1^2)^2} & (3t_1,0,0)
\end{array}
$$

4.2.1 Factorization of Schrödinger operator for $E = 0$

Using the solutions $\phi_{1,r,n}$ and $\phi_{2,r,n}$ defined by (4.13) of Schrödinger equation (4.2) for $E = 0$ we can factor the Schrödinger operators

$$\mathcal{L}_{r,j} = -\partial_{xx} + u_{r,j}, \qquad j = n-1,\ n,\ n+1,$$

as we showed in subsection 1.3.1. For this, we define the logarithmic derivatives $\sigma_{1,r,n} = (\log \phi_{1,r,n})_x$ and $\sigma_{2,r,n} = (\log \phi_{2,r,n})_x$. These functions are, by construction, solutions of the Riccati equation:

$$\sigma_x = u_{r,n} - \sigma^2. \tag{4.21}$$

Futhermore, we can compute Riccati equations for $\sigma_{1,r,n}$ and $\sigma_{2,r,n}$ in terms of the potentials $u_{r,n-1}$ and $u_{r,n+1}$. For, this we keep in mind that these potentials are obtained by means of Darboux–Crum transformations for $u_{r,n}$ as follows:

$$DT(\phi_{1,r,n})u_{r,n} = u_{r,n} - 2(\log\phi_{1,r,n})_{xx} = u_{r,n-1}, \tag{4.22}$$
$$DT(\phi_{2,r,n})u_{r,n} = u_{r,n} - 2(\log\phi_{2,r,n})_{xx} = u_{r,n+1}. \tag{4.23}$$

So, replacing $u_{r,n} = \sigma_{i,r,n}^2 + \sigma_{i,r,n,x}$, $i = 1, 2$, in each of these equations yields to

$$u_{r,n-1} = u_{r,n} - 2(\log\phi_{1,r,n})_{xx} = \sigma_{1,r,n,x} + \sigma_{1,r,n}^2 - 2\sigma_{1,r,n,x} = \sigma_{1,r,n}^2 - \sigma_{1,r,n,x},$$
$$u_{r,n+1} = u_{r,n} - 2(\log\phi_{2,r,n})_{xx} = \sigma_{2,r,n,x} + \sigma_{2,r,n}^2 - 2\sigma_{2,r,n,x} = \sigma_{2,r,n}^2 - \sigma_{2,r,n,x}.$$

This Riccati equations allow us to obtain different factorizations of the Schrödinger operators $\mathcal{L}_{r,n-1}$, $\mathcal{L}_{r,n}$ and $\mathcal{L}_{r,n+1}$, as we show next.

First, we replace σ by $\sigma_{1,r,n}$ and $\sigma_{2,r,n}$ in expression (1.22). We obtain, respectively,

$$(-\partial_x - \sigma_{1,r,n})(\partial_x - \sigma_{1,r,n}) = -\partial_{xx} + \sigma_{1,r,n}^2 + \sigma_{1,r,n,x} = -\partial_{xx} + u_{r,n} = \mathcal{L}_{r,n}, \tag{4.24}$$
$$(-\partial_x - \sigma_{2,r,n})(\partial_x - \sigma_{2,r,n}) = -\partial_{xx} + \sigma_{2,r,n}^2 + \sigma_{2,r,n,x} = -\partial_{xx} + u_{r,n} = \mathcal{L}_{r,n}, \tag{4.25}$$

by Riccati equation (4.21).

Conversely, if we exchange the factors in expressions (4.24) and (4.25) we obtain, respectively,

$$(\partial_x - \sigma_{1,r,n})(-\partial_x - \sigma_{1,r,n}) = -\partial_{xx} + \sigma_{1,r,n}^2 - \sigma_{1,r,n,x} = -\partial_{xx} + u_{r,n-1} = \mathcal{L}_{r,n-1},$$
$$(\partial_x - \sigma_{2,r,n})(-\partial_x - \sigma_{2,r,n}) = -\partial_{xx} + \sigma_{2,r,n}^2 - \sigma_{2,r,n,x} = -\partial_{xx} + u_{r,n+1} = \mathcal{L}_{r,n+1}.$$

And we get the desired factorizations.

Notice that, when performing a Darboux–Crum transformation to potential $u_{r,n}$ with solution $\phi_{1,r,n}$ we arrive to the previous Adler–Moser potential $u_{r,n-1}$ (see equation (4.22)) and when performing a Darboux–Crum transformation with solution $\phi_{2,r,n}$ we arrive to the next Adler–Moser potential $u_{r,n+1}$ (see equation (4.23)), both adjusted to be solutions of KdV$_r$ equation. So, by an inductive argument, the family of Adler–Moser potentials can be directly constructed by means of Darboux–Crum transformations starting from the potential for $n = 0$, i.e., $u_{r,0} = 0$.

4.3 Fundamental matrices for Schrödinger equation with KdV$_r$ potentials. Case $E \neq 0$

In this section, we compute explicitly fundamental matrices of system (4.1) when $u = u_{r,n} = -2(\log\theta_{r,n})_{xx}$ and $E \neq 0$, for $r \geq 1$. We also prove some important properties of these solutions. Recall that $u_{r,n} = u_{r,n}(x, t_r)$ is a solution of KdV$_r$ equation (see Remark 4.3). In this case, the system is:

$$\begin{cases} \Phi_x &= U\Phi = \begin{pmatrix} 0 & 1 \\ u_{r,n} - E & 0 \end{pmatrix}\Phi, \\ \Phi_{t_r} &= V_r\Phi = \begin{pmatrix} -\dfrac{F_{r,x}(u_{r,n})}{2} & F_r(u_{r,n}) \\ (u_{r,n} - E)F_r(u_{r,n}) - \dfrac{F_{r,xx}(u_{r,n})}{2} & \dfrac{F_{r,x}(u_{r,n})}{2} \end{pmatrix}\Phi. \end{cases} \tag{4.26}$$

The zero curvature condition of this system is still the KdV$_r$ equation for $c_i = 0$, $i = 1, \dots, r$:

$$u_{r,n,t_r} = 2f_{r+1,x}(u_{r,n}). \tag{4.27}$$

When $E \neq 0$, we take $\lambda \in \mathbf{C}$ a complex parameter such that $E + \lambda^2 = 0$. Next, we consider the differential systems:

$$\begin{aligned}
Q^+_{n,xx} =& Q^+_{n,x}\left(-2\lambda + 2\frac{\theta_{r,n,x}}{\theta_{r,n}}\right) + Q^+_n\left(2\lambda\frac{\theta_{r,n,x}}{\theta_{r,n}} - \frac{\theta_{r,n,xx}}{\theta_{r,n}}\right), && (4.28)\\
Q^+_{n,t_r} =& Q^+_{n,x}F_r(u_{r,n}) && (4.29)\\
&+ Q^+_n\left(-(-1)^r\lambda^{2r+1} + \lambda F_r(u_{r,n}) + \frac{\theta_{r,n,t_r}}{\theta_{r,n}} - \frac{F_{r,x}(u_{r,n})}{2} - F_r(u_{r,n})\frac{\theta_{r,n,x}}{\theta_{r,n}}\right),\\
Q^-_{n,xx} =& Q^-_{n,x}\left(2\lambda + 2\frac{\theta_{r,n,x}}{\theta_{r,n}}\right) - Q^-_n\left(2\lambda\frac{\theta_{r,n,x}}{\theta_{r,n}} + \frac{\theta_{r,n,xx}}{\theta_{r,n}}\right), && (4.30)\\
Q^-_{n,t_r} =& Q^-_{n,x}F_r(u_{r,n}) && (4.31)\\
&+ Q^-_n\left((-1)^r\lambda^{2r+1} - \lambda F_r(u_{r,n}) + \frac{\theta_{r,n,t_r}}{\theta_{r,n}} - \frac{F_{r,x}(u_{r,n})}{2} - F_r(u_{r,n})\frac{\theta_{r,n,x}}{\theta_{r,n}}\right).
\end{aligned}$$

We have the following relations for solutions of the differential systems (4.28)-(4.29) and (4.30)-(4.31).

Lemma 4.12. *Functions Q^+_n and Q^-_n recursively defined by*

$$Q^+_0 = 1, \qquad Q^+_{n+1} = \frac{\lambda Q^+_n\theta_{r,n+1} + Q^+_{n,x}\theta_{r,n+1} - Q^+_n\theta_{r,n+1,x}}{\theta_{r,n}}, \tag{4.32}$$

$$Q^-_0 = 1, \qquad Q^-_{n+1} = \frac{\lambda Q^-_n\theta_{r,n+1} - Q^-_{n,x}\theta_{r,n+1} + Q^-_n\theta_{r,n+1,x}}{\theta_{r,n}} \tag{4.33}$$

are solutions of the differential systems (4.28)-(4.29) *and* (4.30)-(4.31).

Proof. We prove it by induction on n. For $n = 0$ we have $\theta_{r,0} = 1$, hence, $u_{r,0} = 0$ and $F_r(u_{r,0}) = (-1)^r\lambda^{2r}$. So, $Q^+_0 = 1$ and $Q^-_0 = 1$ are solutions of the systems.

Now, we suppose it true for n and prove it for $n + 1$. We have to prove that expressions

$$\begin{aligned}
Q^+_{n+1} &= \frac{\lambda Q^+_n\theta_{r,n+1} + Q^+_{n,x}\theta_{r,n+1} - Q^+_n\theta_{r,n+1,x}}{\theta_{r,n}},\\
Q^-_{n+1} &= \frac{\lambda Q^-_n\theta_{r,n+1} - Q^-_{n,x}\theta_{r,n+1} + Q^-_n\theta_{r,n+1,x}}{\theta_{r,n}}
\end{aligned}$$

satisfy systems (4.28)-(4.29) and (4.30)-(4.31) respectively, for $n+1$. First, we prove that Q^+_{n+1} satisfies (4.28) and (4.29). By induction hypothesis, we know that Q^+_n

satisfies (4.28). Using this expression and (4.19) we have:

$$
\begin{aligned}
Q^+_{n+1,x} =& \frac{\lambda Q^+_n \theta_{r,n}\theta_{r,n+1,x} + \lambda Q^+_n \theta_{r,n,x}\theta_{r,n+1} - \lambda Q^+_{n,x}\theta_{r,n}\theta_{r,n+1}}{\theta^2_{r,n}} \\
&+ \frac{Q^+_{n,x}\theta_{r,n,x}\theta_{r,n+1} - Q^+_n \theta_{r,n,x}\theta_{r,n+1,x}}{\theta^2_{r,n}}, \\
Q^+_{n+1,xx} =& \frac{Q^+_{n,x}}{\theta^3_{r,n}} p_1(x,t_r,\lambda) + \frac{Q^+_n}{\theta^3_{r,n}} p_2(x,t_r,\lambda),
\end{aligned}
$$

and

$$
Q^+_{n+1,x}\left(-2\lambda + 2\frac{\theta_{r,n+1,x}}{\theta_{r,n+1}}\right) + Q^+_{n+1}\left(2\lambda\frac{\theta_{r,n+1,x}}{\theta_{r,n+1}} - \frac{\theta_{r,n+1,xx}}{\theta_{r,n+1}}\right) = \frac{Q^+_{n,x}}{\theta^3_{r,n}} p_1 + \frac{Q^+_n}{\theta^3_{r,n}} p_2
$$

where

$$
\begin{aligned}
p_1 = p_1(x,t_r,\lambda) &= 2\lambda^2\theta^2_{r,n}\theta_{r,n+1} - 2\lambda\theta_{r,n}\theta_{r,n,x}\theta_{r,n+1} + 2\theta_{r,n}\theta_{r,n,x}\theta_{r,n+1,x} \\
&\quad - \theta^2_{r,n}\theta_{r,n+1,xx}, \\
p_2 = p_2(x,t_r,\lambda) &= -2\lambda^2\theta_{r,n}\theta_{r,n,x}\theta_{r,n+1} + 2\lambda\theta_{r,n}\theta_{r,n,xx}\theta_{r,n+1} + \theta^2_{r,n}\theta_{r,n+1,xx} \\
&\quad - \theta_{r,n}\theta_{r,n,xx}\theta_{r,n+1,x}.
\end{aligned}
$$

Thus, both expressions coincide and Q^+_{n+1} is solution of equation (4.28).

On the other hand, by induction hypothesis, we know that Q^+_n satisfies (4.29). Using this equation, expressions

$$
\begin{aligned}
\sigma_{2,r,n} &= (\log\phi_{2,r,n})_x = \frac{\theta_{r,n+1,x}\theta_{r,n} - \theta_{r,n+1}\theta_{r,n,x}}{\theta_{r,n}\theta_{r,n+1}}, \\
\sigma_{2,r,n,t_r} &= \frac{\theta_{r,n+1,xt_r}}{\theta_{r,n+1}} - \frac{\theta_{r,n,xt_r}}{\theta_{r,n}} + \frac{\theta_{r,n,x}\theta_{r,n,t_r}}{\theta^2_{r,n}} - \frac{\theta_{r,n+1,x}\theta_{r,n+1,t_r}}{\theta^2_{r,n+1}}, \\
Q^+_{n,xt_r} &= Q^+_{n,x}\left(-(-1)^r\lambda^{2r+1} - \lambda F_r(u_{r,n}) + \frac{F_{r,x}(u_{r,n})}{2} + F_r(u_{r,n})\frac{\theta_{r,n,x}}{\theta_{r,n}} + \frac{\theta_{r,n,t_r}}{\theta_{r,n}}\right) \\
&\quad + Q^+_n\left(2\lambda F_r(u_{r,n})\frac{\theta_{r,n,x}}{\theta_{r,n}} + \lambda F_{r,x}(u_{r,n}) - 2F_r(u_{r,n})\frac{\theta_{r,n,xx}}{\theta_{r,n}} + F_r(u_{r,n})\frac{\theta^2_{r,n,x}}{\theta^2_{r,n}}\right. \\
&\quad \left. - \frac{F_{r,xx}(u_{r,n})}{2} - F_{r,x}(u_{r,n})\frac{\theta_{r,n,x}}{\theta_{r,n}} - \frac{\theta_{r,n,x}\theta_{r,n,t_r}}{\theta^2_{r,n}} + \frac{\theta_{r,n,xt_r}}{\theta_{r,n}}\right),
\end{aligned}
$$

the derivative with respect to x of statement 2 of Corollary 3.22 and expression (3.56) for σ_{2,r,n,t_r}, we obtain

$$
Q^+_{n+1,t_r} = Q^+_{n,x}\frac{p_3(x,t_r,\lambda)}{\theta^2_{r,n}} + Q^+_n\frac{p_4(x,t_r,\lambda)}{\theta^2_{r,n}},
$$

where

$$
\begin{aligned}
p_3(x,t_r,\lambda) &= -(-1)^r\lambda^{2r+1}\theta_{r,n}\theta_{r,n+1} + F_r(u_{r,n})\theta_{r,n,x}\theta_{r,n+1} - F_r(u_{r,n})\theta_{r,n}\theta_{r,n+1,x} \\
&\quad + F_{r,x}(u_{r,n})\frac{\theta_{r,n}\theta_{r,n+1}}{2} + \theta_{r,n}\theta_{r,n+1,t_r}, \\
p_4(x,t_r,\lambda) &= -(-1)^r\lambda^{2r+2}\theta_{r,n}\theta_{r,n+1} + (-1)^r\lambda^{2r+1}\theta_{r,n}\theta_{r,n+1,x} + \lambda^2 F_r(u_{r,n})\theta_{r,n}\theta_{r,n+1} \\
&\quad + \lambda^2 P_r\theta_{r,n}\theta_{r,n+1} + \lambda\theta_{r,n}\theta_{r,n+1,t_r} + \lambda F_r(u_{r,n})\theta_{r,n,x}\theta_{r,n+1} \\
&\quad + \lambda F_{r,x}(u_{r,n})\frac{\theta_{r,n}\theta_{r,n+1}}{2} - \lambda F_r(u_{r,n})\theta_{r,n}\theta_{r,n+1,x} + F_{r,x}(u_{r,n})\frac{\theta_{r,n}\theta_{r,n+1,x}}{2} \\
&\quad - P_r\theta_{r,n,x}\theta_{r,n+1,x} - \frac{\theta_{r,n}\theta_{r,n+1,x}\theta_{r,n+1,t_r}}{\theta_{r,n+1}} - F_r(u_{r,n})\theta_{r,n,x}\theta_{r,n+1,x} \\
&\quad + F_r(u_{r,n})\frac{\theta_{r,n}\theta^2_{r,n+1,x}}{\theta_{r,n+1}} + P_r\frac{\theta_{r,n}\theta^2_{r,n+1,x}}{\theta_{r,n+1}} + P_{r,x}\frac{\theta_{r,n}\theta_{r,n+1,x}}{2}.
\end{aligned}
$$

Finally, using relation (4.28) for Q_n^+ and statements 1 and 2 of Corollary 3.22, the right hand side of equation (4.29) for Q_{n+1}^+ reads

$$
\begin{aligned}
&Q^+_{n+1,x}F_r(u_{r,n+1}) + Q^+_{n+1}\left(\lambda^3 + \lambda F_r(u_{r,n+1}) + \tfrac{\theta_{r,n,t_r}}{\theta_{r,n}} - \tfrac{F_{r,x}(u_{r,n+1})}{2} - F_r(u_{r,n+1})\tfrac{\theta_{r,n,x}}{\theta_{r,n}}\right) \\
&\qquad = Q^+_{n,x}\frac{p_3(x,t_r,\lambda)}{\theta^2_{r,n}} + Q^+_n\frac{p_4(x,t_r,\lambda)}{\theta^2_{r,n}}.
\end{aligned}
$$

Therefore, both expressions coincide and Q^+_{n+1} is a solution of equation (4.29).

The proof for Q^-_{n+1} is analogous. □

As a consequence, we have the following result:

Theorem 4.13. *Let n be a non negative integer, then, for $E = -\lambda^2 \neq 0$ and $u = u_{r,n}$, a fundamental matrix for system* (4.26) *is:*

$$
\mathcal{B}^{(r)}_{n,\lambda} = \begin{pmatrix} \phi^+_{r,n} & \phi^-_{r,n} \\ \phi^+_{r,n,x} & \phi^-_{r,n,x} \end{pmatrix}, \tag{4.34}
$$

where

$$
\phi^+_{r,n}(x,t_r,\lambda) = \frac{e^{\lambda x+(-1)^r\lambda^{2r+1}t_r}Q^+_{r,n}(x,t_r,\lambda)}{\theta_{r,n}}, \tag{4.35}
$$

$$
\phi^-_{r,n}(x,t_r,\lambda) = \frac{e^{-\lambda x-(-1)^r\lambda^{2r+1}t_r}Q^-_{r,n}(x,t_r,\lambda)}{\theta_{r,n}}, \tag{4.36}
$$

where $Q^{\pm}_{r,n}$ are functions in x, t_r, λ such that they are solutions of the differential systems (4.28)-(4.29) *and* (4.30)-(4.31).

Proof. We prove it by induction on n. For $n = 0$ the definition $\theta_{r,0} = 1$ leads to $u_{r,0} = 0$. So, system (4.26) becomes

$$
\begin{cases}
\begin{pmatrix} \phi^+_{r,0,x} & \phi^-_{r,0,x} \\ \phi^+_{r,0,xx} & \phi^-_{r,0,xx} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ \lambda^2 & 0 \end{pmatrix}\begin{pmatrix} \phi^+_{r,0} & \phi^-_{r,0} \\ \phi^+_{r,0,x} & \phi^-_{r,0,x} \end{pmatrix}, \\
\begin{pmatrix} \phi^+_{r,0,t_r} & \phi^-_{r,0,t_r} \\ \phi^+_{r,0,xt_r} & \phi^-_{r,0,xt_r} \end{pmatrix} = \begin{pmatrix} 0 & (-1)^r\lambda^{2r} \\ (-1)^r\lambda^{2r+2} & 0 \end{pmatrix}\begin{pmatrix} \phi^+_{r,0} & \phi^-_{r,0} \\ \phi^+_{r,0,x} & \phi^-_{r,0,x} \end{pmatrix}.
\end{cases}
$$

Hence, $\phi^+_{r,0} = e^{\lambda x + (-1)^r \lambda^{2r+1} t_r}$ and $\phi^-_{r,0} = e^{-\lambda x - (-1)^r \lambda^{2r+1} t_r}$ generate $\mathcal{B}^{(r)}_{0,\lambda}$. Since $\theta_{r,0} = 1$, we find $Q^{\pm}_{r,0} = 1$, as in Lemma 4.12.

Next, we suppose it true for n and prove it for $n+1$. Since the functions $\phi^+_{r,n}(x, t_r, \lambda) = e^{\lambda x + (-1)^r \lambda^{2r+1} t_r} \frac{Q^+_{r,n}}{\theta_{r,n}}$, and $\phi^-_{r,n}(x, t_r, \lambda) = e^{-\lambda x - (-1)^r \lambda^{2r+1} t_r} \frac{Q^-_{r,n}}{\theta_{r,n}}$ are solutions of the Schrödinger equation $\phi_{xx} = (u_{r,n} + \lambda^2)\phi$, we apply a Darboux–Crum transformation with $\phi_{2,r,n} = \frac{\theta_{r,n+1}}{\theta_{r,n}}$ to this equation and we obtain:

$$
\begin{aligned}
DT(\phi_{2,r,n}) u_{r,n} &= u_{r,n} - 2(\log \phi_{2,r,n})_{xx} = u_{r,n} - 2\sigma_{2,r,n,x} = u_{r,n+1}, \\
DT(\phi_{2,r,n}) \phi^+_{r,n} &= \phi^+_{r,n,x} - \frac{\phi_{2,r,n,x}}{\phi_{2,r,n}} \phi^+_{r,n} \\
&= \frac{e^{\lambda x + (-1)^r \lambda^{2r+1} t_r}}{\theta_{r,n+1}} \cdot \frac{\lambda Q^+_{r,n}\theta_{r,n+1} + Q^+_{r,n,x}\theta_{r,n+1} - Q^+_{r,n}\theta_{r,n+1,x}}{\theta_{r,n}} \\
&= e^{\lambda x + (-1)^r \lambda^{2r+1} t_r} \frac{Q^+_{r,n+1}}{\theta_{r,n+1}} = \phi^+_{r,n+1}(x, t_r, \lambda), \qquad (4.37) \\
DT(\phi_{2,r,n}) \phi^-_{r,n} &= \phi^-_{r,n,x} - \frac{\phi_{2,r,n,x}}{\phi_{2,r,n}} \phi^-_{r,n} \\
&= \frac{e^{-\lambda x - (-1)^r \lambda^{2r+1} t_r}}{\theta_{r,n+1}} \cdot \frac{-\lambda Q^-_{r,n}\theta_{r,n+1} + Q^-_{r,n,x}\theta_{r,n+1} - Q^-_{r,n}\theta_{r,n+1,x}}{\theta_{r,n}} \\
&= e^{-\lambda x - (-1)^r \lambda^{2r+1} t_r} \frac{(-Q^-_{r,n+1})}{\theta_{r,n+1}} = -\phi^-_{r,n+1}(x, t_r, \lambda), \qquad (4.38)
\end{aligned}
$$

by Lemma 4.12. Hence, $DT(\phi_{2,r,n})\phi^+_{r,n} = \phi^+_{r,n+1}(x, t_r, \lambda)$ and $DT(\phi_{2,r,n})\phi^-_{r,n} = -\phi^-_{r,n+1}(x, t_r, \lambda)$ generate $\mathcal{B}^{(r)}_{n+1,\lambda}$. This ends the proof. $\square$

As far as we know, a general explicit expression for fundamental matrices for system (4.26) has never been computed when $E \neq 0$. As in Theorem 4.7, the key to do that is to control the action of the Darboux–Crum transformations over the differential polynomials F_j, as we showed in Subsection 3.2.1. In Section 4.4 we will give some examples of these fundamental solutions both in the general framework of unadjusted functions τ_i and in the particular adjusted case for $r = 1$, in the same line as in Example 4.11.

Proposition 4.14. *Functions $Q^+_{r,n}, Q^-_{r,n}$ and solutions $\phi^+_{r,n}, \phi^-_{r,n}$ defined in Theorem 4.13 satisfy the relations*

$$
Q^+_{r,n}(x, t_r, -\lambda) = (-1)^n Q^-_{r,n}(x, t_r, \lambda) \quad \text{and} \quad \phi^+_{r,n}(x, t_r, -\lambda) = (-1)^n \phi^-_{r,n}(x, t_r, \lambda).
$$

Proof. We notice that

$$
\phi^+_{r,n}(x, t_r, -\lambda) = e^{-\lambda x - (-1)^r \lambda^{2r+1} t_r} \frac{Q^+_{r,n}(x, t_r, -\lambda)}{\theta_{r,n}},
$$

since $\theta_{r,n}$ does not depend on λ. So, both relations are equivalent and it suffices to prove that $Q^+_{r,n}(x, t_r, -\lambda) = (-1)^n Q^-_{r,n}(x, t_r, \lambda)$. We prove it by induction on n. For $n = 0$, we have that $Q^+_{r,0} = 1 = Q^-_{r,0}$. Hence, $Q^+_{r,0}(x, t_r, -\lambda) = (-1)^0 Q^-_{r,0}(x, t_r, \lambda)$.

Using expresions (4.32) and (4.33), we obtain

$$
\begin{aligned}
&Q^+_{r,n+1}(x,t_r,-\lambda) = \\
&= \frac{-\lambda Q^+_{r,n}(x,t_r,-\lambda)\theta_{r,n+1} + Q^+_{r,n,x}(x,t_r,-\lambda)\theta_{r,n+1} - \theta_{r,n+1,x}Q^+_{r,n}(x,t_r,-\lambda)}{\theta_{r,n}} \\
&= \frac{(-1)^n(-\lambda Q^-_{r,n}(x,t_r,\lambda)\theta_{r,n+1} + Q^-_{r,n,x}(x,t_r,\lambda)\theta_{r,n+1} - \theta_{r,n+1,x}Q^-_{r,n}(x,t_r,\lambda))}{\theta_{r,n}} \\
&= \frac{(-1)^{n+1}(\lambda Q^-_{r,n}(x,t_r,\lambda)\theta_{r,n+1} - Q^-_{r,n,x}(x,t_r,\lambda)\theta_{r,n+1} + \theta_{r,n+1,x}Q^-_{r,n}(x,t_r,\lambda))}{\theta_{r,n}} \\
&= (-1)^{n+1}Q^-_{r,n+1}(x,t_r,\lambda),
\end{aligned}
$$

as we wanted to prove. □

This corollary allows us to compute the determinant of $\mathcal{B}^{(r)}_{n,\lambda}$. First observe that

$$
\begin{aligned}
\det\mathcal{B}^{(r)}_{n,\lambda} &= W(\phi^+_{r,n},\phi^-_{r,n}) = (-1)^n W(\phi^+_{r,n}(x,t_r,\lambda),\phi^+_{r,n}(x,t_r,-\lambda)) \\
&= (-1)^{n+1}\frac{2\lambda Q^+_{r,n}(x,t_r,\lambda)Q^+_{r,n}(x,t_r,-\lambda) + W(Q^+_{r,n}(x,t_r,-\lambda),Q^+_{r,n}(x,t_r,\lambda))}{\theta^2_{r,n}},
\end{aligned}
\tag{4.39}
$$

where $W(\phi_1,\phi_2) = \phi_1\phi_{2,x} - \phi_{1,x}\phi_2$ denotes the Wronskian of ϕ_1 and ϕ_2.

Theorem 4.15. *We have*

$$\det\mathcal{B}^{(r)}_{n,\lambda} = -2\lambda^{2n+1}.$$

Proof. We proceed by induction on n. For $n=0$ we obtain $Q^+_{r,0} = 1$ and $\theta_{r,0} = 1$, so $\det\mathcal{B}^{(r)}_{0,\lambda} = -2\lambda$. Now, we suppose it true for n and prove it for $n+1$. Replacing expression (4.32) for $Q^+_{r,n+1}(x,t_r,\lambda)$ and $Q^+_{r,n+1}(x,t_r,-\lambda)$ in the formula (4.39) and using Proposition 4.14 and the induction hypothesis, we get:

$$\det\mathcal{B}^{(r)}_{n+1,\lambda} = -2\lambda^{2n+3} = -2\lambda^{2(n+1)+1}.$$

As we wanted to prove. □

Remark 4.16. Theorem 4.15 implies that matrix $\mathcal{B}^{(r)}_{n,\lambda}$ is not a fundamental matrix of system (4.10) for $\lambda = E = 0$, since it is not invertible for that value of E. The reason of this is that, by Proposition 4.14, when $\lambda = 0$ we have $\phi^+_{r,n}(x,t_r,0) = (-1)^n\phi^-_{r,n}(x,t_r,0)$, so, both column solutions are linear dependent. We will detail this phenomenon in Section 4.5. In fact, we will show that it is not the same to set $E=0$ in (4.10) and then solve the system, than to solve the system for a generic E and then replace $E=0$ in the solution obtained, i.e., there is not specialization process in this sense.

Remark 4.17. For $n = 0$ and $n = 1$ we have the solutions:

n	$\phi^+_{r,n}$	$\phi^-_{r,n}$
0	$e^{\lambda x+(-1)^r\lambda^{2r+1}t_r}$	$e^{-\lambda x-(-1)^r\lambda^{2r+1}t_r}$
1	$\dfrac{e^{\lambda x+(-1)^r\lambda^{2r+1}t_r}(\lambda x-1)}{x}$	$\dfrac{e^{-\lambda x-(-1)^r\lambda^{2r+1}t_r}(\lambda x+1)}{x}$

In next section we will show a method to compute functions $Q^+_{r,n}$ and $Q^-_{r,n}$ more efficient than solving explicitly equations (4.28), (4.29), (4.30) and (4.31), which will allow us to obtain fundamental matrices $\mathcal{B}^{(r)}_{n,\lambda}$. In particular $\phi^+_{r,1}$ and $\phi^-_{r,1}$ are linearly independent solutions for the Schrödinger operator $-\partial^2 + u_{r,1} - E = 0$, where $u_{r,1} = 2/x^2$ is the constructed rational KdV_r potential, as long as $E \neq 0$.

4.4 Examples of fundamental matrices for the case $E \neq 0$

Along this section we prove that funtions $Q^\pm_n$ defined in Lemma 4.12 satisfy the recursion formula (4.4). This will imply in particular that they are polynomials in x and in some variables $\tau^\pm_i$ with coefficients in $C(\lambda)$. Thus, they generalized the family of Adler–Moser polynomials θ_n.

For the following computations we do not suppose that functions θ_n and $Q^\pm_n$ and potentials u_n are adjusted to any level of the KdV hierarchy.

4.4.1 Generalized Adler–Moser polynomials

In Lemma 4.12 we have obtained the recursive formulas (4.32) and (4.33) for $Q^\pm_n$. As we have seen in the proof of Theorem 4.13, these expressions are obtained applying Darboux–Crum transformations with $\phi_{2,r,n}$ to $\phi^+_{r,n}$ and $\phi^-_{r,n}$ (see expressions (4.37) and (4.38)). For our present discussion, we consider the unadjusted relations given in Lemma 4.12 for the unadjusted polynomials θ_{n-1}, θ_n and θ_{n+1}:

$$Q^+_{n+1} = \frac{\lambda Q^+_n\theta_{n+1} + Q^+_{n,x}\theta_{n+1} - Q^+_n\theta_{n+1,x}}{\theta_n}, \tag{4.40}$$

$$Q^-_{n+1} = \frac{\lambda Q^-_n\theta_{n+1} - Q^-_{n,x}\theta_{n+1} + Q^-_n\theta_{n+1,x}}{\theta_n}. \tag{4.41}$$

If we proceed in the same way performing Darboux–Crum transformations with $\phi_{1,r,n}$ we obtain that functions

$$\begin{aligned} DT(\phi_{1,r,n})\phi^+_{r,n} &= \phi^+_{r,n,x} - \frac{\phi_{1,r,n,x}}{\phi_{1,r,n}}\phi^+_{r,n} \\ &= \frac{e^{\lambda x+(-1)^r\lambda^{2r+1}t_r}}{\theta_{r,n-1}} \cdot \frac{\lambda Q^+_{r,n}\theta_{r,n-1} + Q^+_{r,n,x}\theta_{r,n-1} - \theta_{r,n-1,x}Q^+_{r,n}}{\theta_{r,n}}, \end{aligned}$$

$$\begin{aligned} DT(\phi_{1,r,n})\phi^-_{r,n} &= \phi^-_{r,n,x} - \frac{\phi_{1,r,n,x}}{\phi_{1,r,n}}\phi^-_{r,n} \\ &= \frac{e^{-\lambda x-(-1)^r\lambda^{2r+1}t_r}}{\theta_{r,n-1}} \cdot \frac{-\lambda Q^-_{r,n}\theta_{r,n-1} + Q^-_{r,n,x}\theta_{r,n-1} - \theta_{r,n-1,x}Q^-_{r,n}}{\theta_{r,n}}, \end{aligned}$$

are solutions of the Schrödinger equation for $E \neq 0$ and potential

$$DT(\phi_{1,r,n})u_{r,n} = u_{r,n} - 2(\log\phi_{1,r,n})_{xx} = u_{r,n-1}.$$

In the same way that we did for functions (4.32) and (4.33), we can proof that expressions

$$\begin{aligned} Q^+_{r,n-1} &:= \frac{\lambda Q^+_{r,n}\theta_{r,n-1} + Q^+_{r,n,x}\theta_{r,n-1} - \theta_{r,n-1,x}Q^+_{r,n}}{\lambda^2\theta_{r,n}}, \\ Q^-_{r,n-1} &:= \frac{\lambda Q^-_{r,n}\theta_{r,n-1} - Q^-_{r,n,x}\theta_{r,n-1} + \theta_{r,n-1,x}Q^-_{r,n}}{\lambda^2\theta_{r,n}} \end{aligned}$$

satisfy differential systems (4.28)-(4.29) and (4.30)-(4.31), respectively, for $n-1$. So, we obtain:

$$\begin{aligned} DT(\phi_{1,r,n})\phi^+_{r,n} &= \phi^+_{r,n,x} - \frac{\phi_{1,r,n,x}}{\phi_{1,r,n}}\phi^+_{r,n} = \lambda^2\phi^+_{r,n-1}, \\ DT(\phi_{1,r,n})\phi^-_{r,n} &= \phi^-_{r,n,x} - \frac{\phi_{1,r,n,x}}{\phi_{1,r,n}}\phi^-_{r,n} = -\lambda^2\phi^-_{r,n-1}. \end{aligned}$$

For our present discussion, we need the unadjusted relations for the unadjusted functions θ_{n-1}, θ_n and θ_{n+1}:

$$Q^+_{n-1} = \frac{\lambda Q^+_n\theta_{n-1} + Q^+_{n,x}\theta_{n-1} - \theta_{n-1,x}Q^+_n}{\lambda^2\theta_n}, \tag{4.42}$$

$$Q^-_{n-1} = \frac{\lambda Q^-_n\theta_{n-1} - Q^-_{n,x}\theta_{n-1} + \theta_{n-1,x}Q^-_n}{\lambda^2\theta_n}. \tag{4.43}$$

Notice that, when considering the functions $Q^\pm_n$ defined recursively by (4.40), (4.41) (4.42) and (4.43) by means of the unadjusted polynomials θ_n, they are going to depend on τ_i, $i = 2, \ldots, n$.

Now, we can proof the following result:

Theorem 4.18. *Functions Q^+_n and Q^-_n satisfy the differential recursions:*

$$Q^+_0 = 1, \quad Q^+_1 = \lambda x - 1, \quad Q^+_{n+1,x}Q^+_{n-1} - Q^+_{n+1}Q^+_{n-1,x} = (2n+1){Q^+_n}^2, \tag{4.44}$$

$$Q^-_0 = 1, \quad Q^-_1 = \lambda x + 1, \quad Q^-_{n+1,x}Q^-_{n-1} - Q^-_{n+1}Q^-_{n-1,x} = (2n+1){Q^-_n}^2. \tag{4.45}$$

Proof. In Remark 4.17 we have computed ϕ^+_n and ϕ^-_n for $n = 0$ and 1. We have obtained $Q^\pm_0 = 1$, $Q^+_1 = \lambda x - 1$ and $Q^-_1 = \lambda x + 1$. So, we just have to prove the recursion formulas. First, we prove (4.44). For this, we compute $Q^+_{n+1,x}$ and $Q^+_{n-1,x}$ using expressions (4.40) and (4.42):

$$\begin{aligned} Q^+_{n+1,x} &= \frac{\lambda Q^+_{n,x}\theta_n\theta_{n+1} + \lambda Q^+_n\theta_n\theta_{n+1,x} + Q^+_{n,xx}\theta_n\theta_{n+1} + Q^+_n\theta_{n,x}\theta_{n+1,x}}{\theta_n^2} \\ &- \frac{Q^+_n\theta_n\theta_{n+1,xx} + \lambda Q^+_n\theta_{n,x}\theta_{n+1} + Q^+_{n,x}\theta_{n,x}\theta_{n+1}}{\theta_n^2}, \\ Q^+_{n-1,x} &= \frac{\lambda Q^+_{n,x}\theta_n\theta_{n-1} + \lambda Q^+_n\theta_n\theta_{n-1,x} + Q^+_{n,xx}\theta_n\theta_{n-1} + Q^+_n\theta_{n,x}\theta_{n-1,x}}{\lambda^2\theta_n^2} \\ &- \frac{Q^+_n\theta_n\theta_{n-1,xx} + \lambda Q^+_n\theta_{n,x}\theta_{n-1} + Q^+_{n,x}\theta_{n,x}\theta_{n-1}}{\lambda^2\theta_n^2}. \end{aligned}$$

Replacing this expressions in the recursion formula (4.44) we get:

$$\begin{aligned} Q^+_{n+1,x}Q^+_{n-1} - Q^+_{n+1}Q^+_{n-1,x} &= \frac{Q^{+\,2}_n(\theta_{n+1,xx}\theta_{n-1,x} - \theta_{n+1,x}\theta_{n-1,xx})}{\lambda^2\theta_n^2} \\ &+ \frac{(\lambda^2 Q^+_n + 2\lambda Q^+_n Q^+_{n,x} + Q^+_n Q^+_{n,xx})(\theta_{n+1,x}\theta_{n-1} - \theta_{n+1}\theta_{n-1,x})}{\lambda^2\theta_n^3} \\ &+ \frac{(\lambda Q^{+\,2}_n + Q^+_n Q^+_{n,x})(\theta_{n+1}\theta_{n-1,xx} - \theta_{n+1,xx}\theta_{n-1})}{\lambda^2\theta_n^3}. \end{aligned}$$

We want to compute the expressions for θ_{n+1} and θ_{n-1} in brackets in terms of θ_n. For this, if we derivate with respect to x expression (4.4), we find the second one:

$$\theta_{n+1,xx}\theta_{n-1} - \theta_{n+1}\theta_{n-1,xx} = 2(2n+1)\theta_n\theta_{n,x}. \tag{4.46}$$

In order to compute

$$\theta_{n+1,xx}\theta_{n-1,x} - \theta_{n+1,x}\theta_{n-1,xx} \tag{4.47}$$

we use relation (4.19). We have:

$$\theta_{n+1,xx} = 2\frac{\theta_{n+1,x}\theta_{n,x}}{\theta_n} - \frac{\theta_{n+1}\theta_{n,xx}}{\theta_n} \quad \text{and} \quad \theta_{n-1,xx} = 2\frac{\theta_{n-1,x}\theta_{n,x}}{\theta_n} - \frac{\theta_{n-1}\theta_{n,xx}}{\theta_n}.$$

Replacing both expressions in (4.47) we get:

$$\theta_{n+1,xx}\theta_{n-1,x} - \theta_{n+1,x}\theta_{n-1,xx} = \frac{\theta_{n,xx}}{\theta_n}(\theta_{n+1,x}\theta_{n-1} - \theta_{n+1}\theta_{n-1,x}) = (2n+1)\theta_n\theta_{n,xx}. \tag{4.48}$$

Applying expressions (4.4), (4.46) and (4.48) we arrive to:

$$\begin{aligned} Q^+_{n+1,x}Q^+_{n-1} - Q^+_{n+1}Q^+_{n-1,x} &= (2n+1)\frac{\lambda^2 Q^{+\,2}_n\theta_n - 2\lambda Q^{+\,2}_n\theta_{n,x} + 2\lambda Q^+_n Q^+_{n,x}\theta_n}{\lambda^2\theta_n} \\ &+ (2n+1)\frac{-2Q^+_n Q^+_{n,x}\theta_{n,x} + Q^+_n Q^+_{n,xx}\theta_n + Q^{+\,2}_n\theta_{n,xx}}{\lambda^2\theta_n}. \end{aligned}$$

Finally, differential equation (4.28) for $Q^+_{n,xx}$ yields to:

$$Q^+_{n+1,x}Q^+_{n-1} - Q^+_{n+1}Q^+_{n-1,x} = (2n+1)Q^{+\,2}_n.$$

Analogously, the second recursion formula can be proved. So we have established our result. $\square$

Remark 4.19. When we iterate recurrences (4.44) and (4.45) we will obtain integration constants of x which may depend on λ and $\tau_2, \ldots, \tau_n$. We will denote them by $\tau_2^{\pm}, \ldots, \tau_n^{\pm} \in \mathbf{C}(\lambda, t_r)$. So, the functions $Q_n^{\pm}$ obtained by iterating these recurrences are functions in $x, \tau_2^{\pm}, \ldots, \tau_n^{\pm}$. By Lemma 4.1 for $F = \mathbf{C}[x, \tau_2^{\pm}, \ldots, \tau_n^{\pm}]$ and $a = \lambda$, $b = -1$, we can conclude from this theorem that the functions $Q_n^{\pm}$ are polynomials in $x, \tau_2^{\pm}, \ldots, \tau_n^{\pm}$ with coefficients in $\mathbf{C}(\lambda)$ for all n. Here we write $\mathbf{C}(\lambda)$ to emphasize the dependence on λ. Indeed, their degree as functions of λ is n.

In order to have fundamental solutions of system (4.26), we will have to adjust the $\tau_2^{\pm}, \ldots, \tau_n^{\pm}$ in terms of t_r, in the same way as we did for $\tau_2, \ldots, \tau_n$. We will call *adjusted functions* $Q_{r,n}^{\pm} = Q_{r,n}^{\pm}(x, t_r, \lambda)$, whenever the functions $\tau_2^{\pm}, \ldots, \tau_n^{\pm}$ are fixed to get fundamental matrices $\mathcal{B}_{n,\lambda}^{(r)}$. Otherwise, we will call *unadjusted functions* $Q_n^{\pm} = Q_n^{\pm}(x, t_r, \lambda)$.

Hence, adjusted functions $Q_{r,n}^{\pm}$ are indeed polynomials in x and rational functions in t_r. Thus, Theorems 4.13 and 4.18 determine the algebraic structure of $\phi_{r,n}^{+}$ and $\phi_{r,n}^{-}$. We will show some examples of these adjustements next.

For the first polynomials we find

n	Q_n^+	Q_n^-
0	1	1
1	$\lambda x - 1$	$\lambda x + 1$
2	$\lambda^2 x^3 - 3\lambda x^2 + 3x + \tau_2^+$	$\lambda^2 x^3 + 3\lambda x^2 + 3x + \tau_2^-$
3	Q_3^+	Q_3^-

where

$$\begin{aligned} Q_3^+ &= \lambda^3 x^6 - 6\lambda^2 x^5 + 15\lambda x^4 - 15x^3 + 5\lambda x^3 \tau_2^+ - 15x^2\tau_2^+ \\ &\quad - (\lambda\tau_3^+ + 5(\tau_2^+)^2)x + \tau_3^+, \\ Q_3^- &= \lambda^3 x^6 + 6\lambda^2 x^5 + 15\lambda x^4 + 15x^3 + 5\lambda x^3 \tau_2^- + 15x^2\tau_2^- \\ &\quad + (\lambda\tau_3^- + 5(\tau_2^-)^2)x + \tau_3^-. \end{aligned} \tag{4.49}$$

4.4.2 Examples of fundamental matrices for the case $E \neq 0$

We can compute fundamental matrices for system (4.26) for any n using recursion formulas (4.44) and (4.45). We present here some explicit computations using SAGE for these fundamental solutions when $E = -\lambda^2 \neq 0$ and $n = 0, 1, 2$ and 3, for same potentials as in Example 4.11.

1. We first expose examples of fundamental solutions for unadjusted functions θ_n and $Q_n^{\pm}$:

$\phi_{r,n}^{+}$	$\phi_{r,n}^{-}$
$e^{\lambda x + (-1)^r \lambda^{2r+1} t_r}$	$e^{-\lambda x - (-1)^r \lambda^{2r+1} t_r}$
$\dfrac{e^{\lambda x + (-1)^r \lambda^{2r+1} t_r}(\lambda x - 1)}{x}$	$\dfrac{e^{-\lambda x - (-1)^r \lambda^{2r+1} t_r}(\lambda x + 1)}{x}$

$$\frac{e^{\lambda x+(-1)^r\lambda^{2r+1}t_r}(\lambda^2x^3-3\lambda x^2+3x+\tau_2^+)}{x^3+\tau_2} \quad \frac{e^{-\lambda x-(-1)^r\lambda^{2r+1}t_r}(\lambda^2x^3+3\lambda x^2+3x+\tau_2^-)}{x^3+\tau_2}$$

$$\frac{e^{\lambda x+(-1)^r\lambda^{2r+1}t_r}Q_3^+(\lambda,x,\tau_2^+,\tau_3^+)}{x^6+5x^3\tau_2+x\tau_3-5\tau_2^2} \quad \frac{e^{-\lambda x-(-1)^r\lambda^{2r+1}t_r}Q_3^-(\lambda,x,\tau_2^-,\tau_3^-)}{x^6+5x^3\tau_2+x\tau_3-5\tau_2^2}$$

where Q_3^+ and Q_3^- are the polynomials given in (4.49).

2. Next, we expose fundamental solutions for potentials which are solutions of the first level of the KdV hierarchy, KdV$_1$ equation: $u_{t_1}=\frac{3}{2}uu_x-\frac{1}{4}u_{xxx}$. We also show the explicit choice of the functions $\tau_i^{\pm}$. The choice of functions τ_i is the same as in Example 4.11.

$\phi_{1,n}^+$	$\phi_{1,n}^-$	$(\tau_2^{\pm},\dots,\tau_n^{\pm})$
$e^{\lambda x-\lambda^3t_1}$	$e^{-\lambda x+\lambda^3t_1}$	
$\frac{e^{\lambda x-\lambda^3t_1}(\lambda x-1)}{x}$	$\frac{e^{-\lambda x+\lambda^3t_1}(\lambda x+1)}{x}$	
$\frac{e^{\lambda x-\lambda^3t_1}(\lambda^2x^3-3\lambda x^2+3x+3\lambda^2t_1)}{x^3+3t_1}$	$\frac{e^{-\lambda x+\lambda^3t_1}(\lambda^2x^3+3\lambda x^2+3x+3\lambda^2t_1)}{x^3+3t_1}$	$(3\lambda^2t_1)$
$\frac{e^{\lambda x-\lambda^3t_1}Q_3^+(\lambda,x,t_1)}{x^6+15x^3t_1-45t_1^2}$	$\frac{e^{-\lambda x+\lambda^3t_1}Q_3^-(\lambda,x,t_1)}{x^6+15x^3t_1-45t_1^2}$	$(3\lambda^2t_1,-45(\lambda^3t_1^2\pm t_1))$

where

$$\begin{aligned} Q_3^+(\lambda,x,t_1)=&\lambda^3x^6-6\lambda^2x^5+15\lambda x^4-15x^3+15\lambda^3x^3t_1-45\lambda^2x^2t_1+45\lambda xt_1\\ &-45\lambda^3t_1^2-45t_1,\\ Q_3^-(\lambda,x,t_1)=&\lambda^3x^6+6\lambda^2x^5+15\lambda x^4+15x^3+15\lambda^3x^3t_1+45\lambda^2x^2t_1+45\lambda xt_1\\ &-45\lambda^3t_1^2+45t_1. \end{aligned}$$

Notice that the adjustment of the functions $(\tau_2^{\pm},\dots,\tau_n^{\pm})$ is not linear neither in t_1 nor in λ.

4.5 Spectral curves and KdV hierarchy in $1+1$ dimensions

In this section we will show how the points of the spectral curves for the stationary setting explained in 4.1.1 are related with the solutions of the Schrödinger operator with rational potentials in the $1+1$ KdV hierarchy.

Recall that the rational soliton $u_{r,n}$ restricted to $t_r=0$ is the well known n-soliton $u_n^{(0)}(x)=n(n+1)x^{-2}$ (Lemma 4.5). Let Γ_n be its affine spectral curve. This complex plane curve has as defining equation

$$p_n(E,\mu)=\mu^2-E^{2n+1}.$$

Our goal along this chapter was to obtain the algebraic structure of a fundamental matrix for the Schrödinger operator $-\partial_x^2 + u_{r,n} - E$ by means of the system (4.1). For this purpose we needed to use a parametric representation of the spectral curve Γ_n. Observe that Γ_n is a rational singular plane curve, nevertheless we can have a global parametrization in the sense given in [12]. In fact, we have taken the parametrization:

$$\chi(\lambda) = \left(-\lambda^2,\ i\lambda^{2n+1}\right)$$

and then $E = -\lambda^2$ as was taken since Section 4.3. Observe that the unique affine singular point of the spectral curve is reached for $\lambda = 0$. Hence, whenever $\lambda \neq 0$ we obtain regular points on Γ_n and we can obtain the desired description of the fundamental matrix $\mathcal{B}^{(r)}_{n,\lambda}$ as it is given in Theorem 4.13. On the other hand, at the singular point $\chi(0) = (0,0)$ the fundamental matrix for the system (4.1) must be obtained in a specific way, see Theorem 4.7.

The fundamental solutions $\phi_{1,r,n}(x,t_r)$ and $\phi_{2,r,n}(x,t_r)$ obtained in Theorem 4.7 were used as source to perform Darboux–Crum transformations. In particular, for $t_r = 0$, we get the functions:

$$\phi^{(0)}_{1,n}(x) = \phi_{1,r,n}(x, t_r = 0), \qquad \phi^{(0)}_{2,n}(x) = \phi_{2,r,n}(x, t_r = 0)$$

and the corresponding potentials are transformed as is suggested in the following diagram:

$$\begin{array}{ccccc} u^{(0)}_{n-1} & \xleftarrow{DT(\phi^{(0)}_{1,n})} & u^{(0)}_n & \xrightarrow{DT(\phi^{(0)}_{2,n})} & u^{(0)}_{n+1} \\ \downarrow & & \downarrow & & \downarrow \\ \Gamma_{n-1} & & \Gamma_n & & \Gamma_{n+1} \end{array} \tag{4.50}$$

This situation is a particular case of a more general one that has been obtained in Theorem 3.16. The diagram (4.50) has its time dependent counterpart (see (4.22) and (4.23)):

$$u_{r,n-1} \xleftarrow{DT(\phi_{1,r,n})} u_{r,n} \xrightarrow{DT(\phi_{2,r,n})} u_{r,n+1} \tag{4.51}$$

The fundamental matrix $\mathcal{B}^{(r)}_{n,0}$ associated to the functions $\phi_{1,r,n}$ and $\phi_{2,r,n}$ can not be changed by the same Darboux–Crum transformations used for the potentials since there is a loss of independent solutions; in fact we have the following diagram

$$\begin{array}{ccccc} & & \phi_{1,r,n} & \xrightarrow{DT(\phi_{2,r,n})} & \phi_{1,r,n+1} \\ \phi_{2,r,n-1} & \xleftarrow{DT(\phi_{1,r,n})} & \phi_{2,r,n} & & \end{array} \tag{4.52}$$

On the other hand, whenever the point on the spectral curve is a regular point, that is $\lambda \neq 0$, we have obtained the behaviour of the fundamental matrices $\mathcal{B}^{(r)}_{j,\lambda}$, $j = n-1,\ n,\ n+1$, as it is encoded in the following diagram:

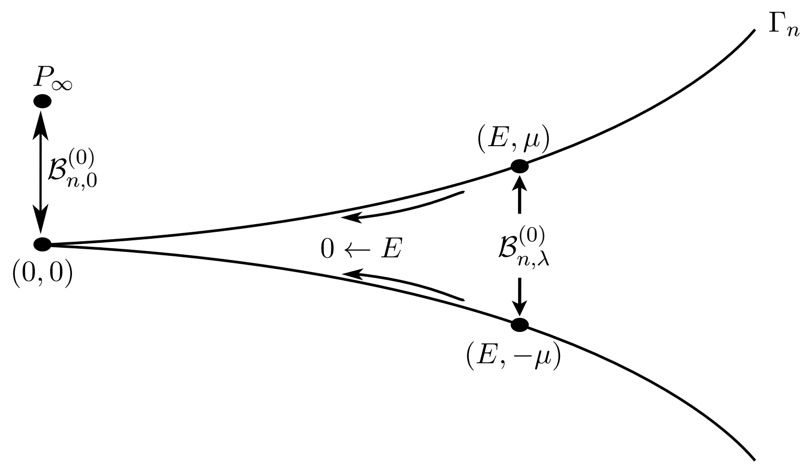
$$\begin{array}{ccccc} \phi^+_{r,n-1} & \xleftarrow{DT(\phi_{1,r,n})} & \phi^+_{r,n} & \xrightarrow{DT(\phi_{2,r,n})} & \phi^+_{r,n+1} \\ \phi^-_{r,n-1} & \xleftarrow{DT(\phi_{1,r,n})} & \phi^-_{r,n} & \xrightarrow{DT(\phi_{2,r,n})} & \phi^-_{r,n+1} \end{array} \tag{4.53}$$

All these situations are reflected in the time dependent frame coming from the stationary one, as we have seen. In particular, in the lack of specialization process from $\mathcal{B}^{(r)}_{n,\lambda}$ to $\mathcal{B}^{(r)}_{n,0}$. According to Theorem 4.15, we have that $\det \mathcal{B}^{(r)}_{n,\lambda} = -2\lambda^{2n+1}$, whereas we have $\det \mathcal{B}^{(r)}_{n,0} = 2n+1$.

Remark 4.20. As we explained in Remark 3.2, given two fundamental solutions of the Schrödinger equation (4.2) for the same value of E and stationary potential $u^{(0)}$, each of them corresponds to a point of the pair of conjugated points of the spectral curve $\{(E,\mu),(E,-\mu)\}$.

In our case, for each $E \neq 0$ we have the two fundamental solutions $(\phi_n^+)^{(0)}$ and $(\phi_n^-)^{(0)}$ corresponding to the couple of conjugated regular points $\{(E,\mu),(E,-\mu)\}$ of Γ_n. However, when $E=0$, we have the two fundamental solutions $\phi^{(0)}_{1,n}$ and $\phi^{(0)}_{2,n}$, corresponding to the points $O=(0,0)\in\Gamma_n$ and $P_\infty=[0:1:0]\in\overline{\Gamma}_n$ respectively, as we have seen. Thus, the point P_∞ appears as "conjugated" point of the origin of Γ_n.

The fundamental matrix for each value of E pairs the solutions for both conjugated points as we illustrate in the following diagram:



From the diagram, we see that when $E=0$, both solutions $(\phi_n^+)^{(0)}$ and $(\phi_n^-)^{(0)}$ collapse to $\phi^{(0)}_{1,n}$, but the solution $\phi^{(0)}_{2,n}$, which can not be detected from the affine situation for $(\phi_n^+)^{(0)}$ and $(\phi_n^-)^{(0)}$, appears.

Next we have computed an explicit example to illustrate the relationship between spectral curves and KdV hierarchy in $1+1$ dimensions for rational solitons.

Example 4.21. Consider the case $r=1$ and $n=2$. Let $u_{1,2}(x,t_1)=\frac{6x(x^3-6t_1)}{(x^3+3t_1)^2}$ be the KdV$_1$ rational soliton obtained by taking $(\tau_2,\tau_3)=(3t_1,0)$. Then, the corresponding stationary potential is given by $u_2^{(0)}(x)=u_{1,2}(x,t_1=0)=6x^{-2}$ (see Lemma 4.5). Its spectral curve is $\Gamma_2: p_2(E,\mu)=\mu^2-E^5$.

Futhermore, the stationary Schrödinger operator presents two types of solutions a priori. When $E=0$, the solutions are

$$\phi_{1,2}^{(0)} := \phi_{1,1,2}(x,t_1=0) = x^{-2} \qquad \text{and} \qquad \phi_{2,2}^{(0)} := \phi_{2,1,2}(x,t_1=0) = x^3. \tag{4.54}$$

where

$$\phi_{1,1,2}(x,t_1) = \frac{x}{x^3+3t_1} \qquad \text{and} \qquad \phi_{2,1,2}(x,t_1) = \frac{x^6+15x^3t_1-45t_1^2}{x^3+3t_1}$$

as they were computed in Example 4.11. In this case, we have the following diagram:

$$\begin{array}{ccccc} u_1^{(0)} = 2/x^2 & \xleftarrow{DT(\phi_{1,2}^{(0)})} & u_2^{(0)} = 6/x^2 & \xrightarrow{DT(\phi_{2,2}^{(0)})} & u_3^{(0)} = 12/x^2 \\ \downarrow & & \downarrow & & \downarrow \\ \mu^2 - E^3 = 0 & & \mu^2 - E^5 = 0 & & \mu^2 - E^7 = 0 \end{array} \tag{4.55}$$

When energy $E \neq 0$, in Section 4.4.2 we have computed the solutions

$$\phi_{1,2}^{+} = \frac{e^{\lambda x-\lambda^3 t_1}(\lambda^2x^3-3\lambda x^2+3x+3\lambda^2t_1)}{x^3+3t_1}, \quad \phi_{1,2}^{-} = \frac{e^{-\lambda x+\lambda^3 t_1}(\lambda^2x^3+3\lambda x^2+3x+3\lambda^2t_1)}{x^3+3t_1} \tag{4.56}$$

where we have adjusted parameters $\tau_2^+ = 3\lambda^2 t_1 = \tau_2^-$. Next, we take $t_1=0$ to obtain

$$\begin{aligned} \phi_2^{+(0)}(x,\lambda) = \phi_{1,2}^{+}(x,t_1=0,\lambda) &= \frac{e^{\lambda x}(\lambda^2x^3-3\lambda x^2+3x)}{x^3}, \\ \phi_2^{-(0)}(x,\lambda) = \phi_{1,2}^{-}(x,t_1=0,\lambda) &= \frac{e^{-\lambda x}(\lambda^2x^3+3\lambda x^2+3x)}{x^3}. \end{aligned}$$

These functions are solutions of the Schrödinger operator for the stationary potential $u_2^{(0)} = 6/x^2$ whenever $E \neq 0$. Observe that $\phi_2^{+(0)}(x,0) = 3/x^2 = \phi_2^{-(0)}(x,0)$, and then they are no longer independent (see 4.17 for the general case).

Next, we will show how the Darboux–Crum transformations act on time dependent potentials and solutions. First recall that for any potential u, we have defined its Darboux–Crum transformation as

$$DT(\phi_{i,r,n})u = u - 2\left(\log \phi_{i,r,n}\right)_{xx}, \qquad i=1,2.$$

Next, we perform the Darboux–Crum transformations by means of $\phi_{1,1,2}$ and $\phi_{2,1,2}$ to our initial potential $u_{1,2}$. In these cases we have obtained

$$u_{1,1} = \frac{2}{x^2} \xleftarrow{DT(\phi_{1,1,2})} u_{1,2} = \frac{6x(x^3-6t_1)}{(x^3+3t_1)^2} \xrightarrow{DT(\phi_{2,1,2})} u_{1,3} = \frac{6x(2x^9+675x^3t_1^2+1350t_1^3)}{(x^6+15x^3t_1-45t_1^2)^2}$$

Then, we must consider the Schrödinger operators

$$-\partial_x^2 + u_{1,j}(x,t_1) - E, \qquad j=1,2,3.$$

Their solutions $\phi_{1,j}^{+}$ and $\phi_{1,j}^{-}$ were given in Section 4.4.2.

It should be noted that if the energy is not zero, these solutions inherit the same behaviour as their corresponding potentials when the Darboux–Crum transformations $DT(\phi_{1,1,2})$ and $DT(\phi_{2,1,2})$ act on them. Hence we obtain the following diagram

$$\phi^+_{1,1} = \frac{e^{\lambda x-\lambda^3 t_1}(\lambda x-1)}{x} \xleftarrow{DT(\phi_{1,1,2})} \phi^+_{1,2} \xrightarrow{DT(\phi_{2,1,2})} \phi^+_{1,3} = \frac{e^{\lambda x-\lambda^3 t_1}Q^+_3(\lambda,x,t_1)}{x^6+15x^3t_1-45t_1^2}$$

$$\phi^-_{1,1} = \frac{e^{-\lambda x+\lambda^3 t_1}(\lambda x+1)}{x} \xleftarrow{DT(\phi_{1,1,2})} \phi^-_{1,2} \xrightarrow{DT(\phi_{2,1,2})} \phi^-_{1,3} = \frac{e^{-\lambda x+\lambda^3 t_1}Q^-_3(\lambda,x,t_1)}{x^6+15x^3t_1-45t_1^2}$$

where the solutions $\phi^+_{1,2}$ and $\phi^-_{1,2}$ are given by (4.56) and

$$\begin{aligned} Q^+_3(x,t_1,\lambda) =& \lambda^3x^6 - 6\lambda^2x^5 + 15\lambda x^4 - 15x^3 + 15\lambda^3x^3t_1 - 45\lambda^2x^2t_1 + 45\lambda x t_1 \\ &- 45\lambda^3t_1^2 - 45t_1, \\ Q^-_3(x,t_1,\lambda) =& \lambda^3x^6 + 6\lambda^2x^5 + 15\lambda x^4 + 15x^3 + 15\lambda^3x^3t_1 + 45\lambda^2x^2t_1 \\ &+ 45\lambda x t_1 - 45\lambda^3t_1^2 + 45t_1. \end{aligned}$$

The zero energy case is essentially different from the Darboux–Crum transformations point of view. We can only partially obtain the previous diagram,

$$\phi_{1,1,2} = \frac{x}{x^3+3t_1} \xrightarrow{DT(\phi_{2,1,2})} \phi_{1,1,3} = \frac{x^3+3t_1}{x^6+15x^3t_1-45t_1^2}$$

$$\phi_{2,1,1} = \frac{x^3+3t_1}{x} \xleftarrow{DT(\phi_{1,1,2})} \phi_{2,1,2} = \frac{x^6+15x^3t_1-45t_1^2}{x^3+3t_1}$$

To compute fundamental matrices associated to $u_{1,1}$ and $u_{1,3}$ we have to use Theorem 4.7 (see Example 4.11).

4.6 Differential Galois groups

In this section we study the Picard–Vessiot extensions given by the fundamental matrices $\mathcal{B}^{(r)}_{n,0}$ and $\mathcal{B}^{(r)}_{n,\lambda}$ for the differential systems (4.10) and (4.26), obtained for energy levels $E=0$ and $E\neq 0$ respectively. We compute their corresponding Galois group, say $\mathcal{G}^{(r)}_{n,0}$ and $\mathcal{G}^{(r)}_{n,\lambda}$ respectively. We recall that the field of coefficients of our systems is the differential field $K_r = \mathbf{C}(x,t_r)$ with field of constants $\mathbf{C}$.

We point out that the behaviour that both the Picard–Vessiot extensions and the Galois groups present depends strongly on the affine point $P=(E,\mu)$ of the corresponding spectral curve. The Galois groups will be isomorphic when the point $P=(E,\mu)$ is a regular point of Γ_n. However, when the point is a singular one, i.e., $E=0$, there will not be differential extension.

4.6.1 Case $E = 0$

For this case we have the fundamental matrix

$$\mathcal{B}^{(r)}_{n,0} = \begin{pmatrix} \phi_{1,r,n} & \phi_{2,r,n} \\ \phi_{1,r,n,x} & \phi_{2,r,n,x} \end{pmatrix},$$

where $\phi_{1,r,n}, \phi_{1,r,n,x}, \phi_{2,r,n}, \phi_{2,r,n,x}$ are rational functions in x and t_r, hence they are in K_r. So, the Picard-Vessiot field is again K_r, i.e., there is no differential extension. Thus, the differential Galois group is the trivial group, $\mathcal{G}^{(r)}_{n,0} = \{\mathrm{id}_2\}$.

4.6.2 Case $E \neq 0$

In this case, we compute the differential extension given for each value of $\lambda \neq 0$. For this, we fix a value of λ different from zero, $\lambda = \lambda_0$, then the point $P = (E_0, \mu_0)$ is a regular point of Γ_n, that is $E_0 \neq 0$. The fundamental matrix is

$$\mathcal{B}^{(r)}_{n,\lambda_0} = \begin{pmatrix} \phi^+_{r,n}(\lambda_0) & \phi^-_{r,n}(\lambda_0) \\ \phi^+_{r,n,x}(\lambda_0) & \phi^-_{r,n,x}(\lambda_0) \end{pmatrix},$$

for $\phi^+_{r,n}(\lambda_0)$, $\phi^+_{r,n,x}(\lambda_0)$, $\phi^-_{r,n}(\lambda_0)$ and $\phi^-_{r,n,x}(\lambda_0) \in K_r(\eta_r)$, with $\eta_r = e^{\lambda_0 x + (-1)^r \lambda_0^{2r+1} t_r}$. Then, the Picard–Vessiot field is $L_r = K_r(\eta_r)$.

To compute the differential Galois group $\mathcal{G}^{(r)}_{n,\lambda_0}$ in this case, we just have to compute the action of $\mathcal{G}^{(r)}_{n,\lambda_0}$ on η_r. For this, let σ in $\mathcal{G}^{(r)}_{n,\lambda_0}$ be an automorphism of the differential Galois group, then

$$\begin{aligned} \left(\frac{\sigma(\eta_r)}{\eta_r}\right)_x &= \frac{\sigma(\lambda_0 \eta_r) - \lambda_0 \sigma(\eta_r)}{\eta_r} = \frac{\lambda_0 \sigma(\eta_r) - \lambda_0 \sigma(\eta_r)}{\eta_r} = 0, \\ \left(\frac{\sigma(\eta_r)}{\eta_r}\right)_{t_r} &= \frac{\sigma((-1)^r \lambda_0^{2r+1} \eta_r) - (-1)^r \lambda_0^{2r+1} \sigma(\eta_r)}{\eta_r} \\ &= \frac{(-1)^r \lambda_0^{2r+1} \sigma(\eta_r) - (-1)^r \lambda_0^{2r+1} \sigma(\eta_r)}{\eta_r} = 0. \end{aligned}$$

Therefore $\frac{\sigma(\eta_r)}{\eta_r}$ is a constant in K_r. Hence $\sigma(\eta_r) = c \cdot \eta_r$ for some $c \in \mathbf{C}$. As a consequence we get that, for each λ_0 and every n, the differential Galois group is isomorphic to the multiplicative group, say

$$\mathcal{G}^{(r)}_{n,\lambda_0} \simeq G_m = \left\{ \begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} : c \in \mathbf{C}^* \right\}.$$

Remark 4.22. Since the Galois groups $\mathcal{G}^{(r)}_{n,\lambda_0}$ are obtained for a particular value of λ by especialization process, they do not depend on λ. For a spectral study of the Picard–Vessiot extensions see [71].

Recall that the fundamental matrix $\mathcal{B}^{(r)}_{n,\lambda_0}$ is obtained by means of Darboux–Crum transformations of the fundamental matrix $\mathcal{B}^{(r)}_{n-1,\lambda_0}$ (see (4.37) and (4.38)). Then, the isomorphism among the Galois groups $\mathcal{G}^{(r)}_{n,\lambda_0}$, for all n, implies that the Galois group is invariant under Darboux–Crum transformations, as expected by Theorem 2.2.

4.6.3 Global behaviour of the differential Galois groups

Let consider the family of linear algebraic groups $\left\{ \mathcal{G}^{(r)}_{n,\lambda} \right\}_{\lambda\in\mathbf{C}}$. Then for each point in Γ_n we have found a linear algebraic group. As a result of our constructions we have a sheave structure of groups on the regular points of Γ_n

$$\Gamma_n^* = \Gamma_n \setminus \mathrm{Sing}(\Gamma_n) \ni (-\lambda^2, i\lambda^{2n+1}) \longrightarrow \mathcal{G}^{(r)}_{n,\lambda}.$$

For each $\lambda \in \mathbf{C}$, the situation is encoded in the following diagram

$$\begin{tikzcd}
\mathcal{G}^{(r)}_{n-1,\lambda} \arrow[d] & \mathcal{G}^{(r)}_{n,\lambda} \arrow[d] & \mathcal{G}^{(r)}_{n+1,\lambda} \arrow[d] \\
\Gamma^*_{n-1} \arrow[r, "\text{Blowing-up}"] & \Gamma^*_n \arrow[r, "\text{Blowing-up}"] & \Gamma^*_{n+1} \\
\mathcal{L}_{n-1} \arrow[u, dotted] & \mathcal{L}_n \arrow[u, dotted] \arrow[l, "DT(\phi^{(0)}_{1,n})"'] \arrow[r, "DT(\phi^{(0)}_{2,n})"] & \mathcal{L}_{n+1} \arrow[u, dotted]
\end{tikzcd} \tag{4.57}$$

We observe *the invariance of the Galois group with respect to:*

- *Time (i.e., it is invariant by the flow of the KdV equation).*
- *Generic values of the spectral parameter, i.e., moving along the regular points of the spectral curve.*
- *Darboux transformations.*

Although this invariant behaviour of the Galois group is proved here for the rational solutions of Adler–Moser type, we conjecture that it is also true for arbitrary algebro-geometric solutions of the KdV_r equation, i.e., for solutions associated to spectral curves different from $\mu^2 - E^{2n+1} = 0$.

Chapter 5

Generalized Adler–Moser potentials

In this chapter we present a family of parameterized KdV potentials which extends, in some sense we will explain later, the family of potentials computed by Adler and Moser in [7]. As we will see, these potentials are rational functions in x, t_r and in a parameter λ.

The work exposed here is based on a joint work with Sonia Jiménez, Juan J. Morales and María Ángeles Zurro.

Consider the derivations ∂_x, $\partial_{t_1}, \partial_{t_2}, \ldots, \partial_{t_m}$ with respect to the variables x and $t_1, \ldots, t_m$. Along this chapter K_r will denote the differential field of rational functions in the variables x and t_r with compatible derivations ∂_x and ∂_{t_r}, for $r = 1, \ldots, m$, and with field of constants $\mathbf{C}$ algebraically closed and of characteristic zero. Let $E, \lambda \in \mathbf{C}$ be two parameters related by $E + \lambda^2 = 0$.

Consider the unadjusted functions Q_n^+ and Q_n^- as introduced in Section 4.4. They are defined by the differential recursions (4.44) and (4.45). For the reader's convenience, we reproduce those equations:

$$Q_0^+ = 1, \quad Q_1^+ = \lambda x - 1, \quad Q_{n+1,x}^+ Q_{n-1}^+ - Q_{n+1}^+ Q_{n-1,x}^+ = (2n+1) Q_n^{+\,2}. \tag{5.1}$$
$$Q_0^- = 1, \quad Q_1^- = \lambda x + 1, \quad Q_{n+1,x}^- Q_{n-1}^- - Q_{n+1}^- Q_{n-1,x}^- = (2n+1) Q_n^{-\,2}. \tag{5.2}$$

Recall that when these functions are unadjusted they are polynomials in x and in some integration constants of x, $\tau_i^{\pm} \in \mathbf{C}(t_r)$, for $i = 2, \ldots, n$, with coefficients in $\mathbf{C}(\lambda)$ (see Remark 4.19). Here, we write $\mathbf{C}(\lambda)$ to emphasize the dependence on λ. In this chapter, we consider the functions:

$$u_n^+(x, \tau_2^+, \ldots, \tau_n^+, \lambda) := -2(\log Q_n^+)_{xx}, \tag{5.3}$$
$$u_n^-(x, \tau_2^-, \ldots, \tau_n^-, \lambda) := -2(\log Q_n^-)_{xx}. \tag{5.4}$$

These functions are, obviously, rational functions in $x, \tau_2^\pm, \dots, \tau_n^\pm$ and λ, i.e., they belong to K_r.

Later we will see that these expressions come up in a natural way as Darboux–Crum transformations of the unadjusted Adler–Moser potentials u_n (see equation (5.7)) and that in fact, there exists a unique adjustment of $\tau_2^\pm, \dots, \tau_n^\pm$ for which they are KdV potentials as well (see Theorem 5.12).

Example 5.1. We compute next some examples of this functions:

n	u_n^+	u_n^-
0	0	0
1	$\frac{2\lambda^2}{(\lambda x-1)^2}$	$\frac{2\lambda^2}{(\lambda x+1)^2}$
2	$\frac{6(\lambda x-1)(\lambda^3x^3-3\lambda^2x^2+3\lambda x-2\lambda\tau_2^+-3)}{(\lambda^2x^3-3\lambda x^2+3x+\tau_2^+)^2}$	$\frac{6(\lambda x+1)(\lambda^3x^3+3\lambda^2x^2+3\lambda x-2\lambda\tau_2^-+3)}{(\lambda^2x^3+3\lambda x^2+3x+\tau_2^-)^2}$

Remark 5.2. Taking $\lambda = 0$ in recurrences (5.1) and (5.2) we get for the first polynomials Q_n^+ and Q_n^-:

$$Q_0^+ = 1, \quad Q_1^+ = -1, \quad Q_2^+ = 3x + \tau_2^+, \quad Q_3^+ = -15x^3 - 15x^2\tau_2^+ - 5x(\tau_2^+)^2 + \tau_3^+.$$
$$Q_0^- = 1, \quad Q_1^- = 1, \quad Q_2^- = 3x + \tau_2^-, \quad Q_3^- = 15x^3 + 15x^2\tau_2^+ + 5x(\tau_2^+)^2 + \tau_3^+.$$

With the choice $\tau_2^\pm = 0$, $\tau_3^+ = -15\tau_2$ and $\tau_3^- = 15\tau_2$, where τ_2 is the integration constant obtained for the Adler–Moser polynomial θ_2, we recover the first Adler–Moser polynomials up to multiplication by the constants $c_n^+ = (-1)^n \prod_{i=1}^n (2i-1)$ for Q_n^+ and $c_n^- = \prod_{i=1}^n (2i-1)$ for Q_n^-, $n \geq 1$, i.e., we obtain:

$$Q_1^+ = -1 = -\theta_0, \qquad Q_2^+ = 3x = 3\theta_1, \qquad Q_3^+ = -15x^3 + 15\tau_2 = -15\theta_2.$$
$$Q_1^- = 1 = \theta_0, \qquad Q_2^- = 3x = 3\theta_1, \qquad Q_3^- = 15x^3 + 15\tau_2 = 15\theta_2.$$

Iterating each recurence equation with these initial values we recover the family of Adler–Moser polynomials multiplied by c_n^+ and c_n^- respectively. In the same way, we can recover the family of Adler–Moser potentials. So, the family of Adler–Moser potentials is contained in this family of λ-parameterized functions. In this sense, we say that this family extends the one of Adler and Moser, and we will call the functions u_n^+ and u_n^- *generalized Adler–Moser potentials*.

5.1 Fundamental matrices for Schrödinger operator with generalized Adler–Moser potentials

In this section, first we consider the Schrödinger operator

$$\mathcal{L}_n = -\partial_{xx} + u_n$$

for the unadjusted Adler–Moser potentials u_n defined in (4.6). Since in this case there are no conditions for t_r, by Theorems 4.7 and 4.13 we know that the unadjusted functions

$$\phi_{1,n} = \frac{\theta_{n-1}}{\theta_n} \qquad \text{and} \qquad \phi_{2,n} = \frac{\theta_{n+1}}{\theta_n} \tag{5.5}$$

are solutions of the Schrödinger equation $\mathcal{L}_n\phi = 0$, and that the unadjusted functions

$$\phi_n^+ = \frac{e^{\lambda x+(-1)^r\lambda^{2r+1}t_r}Q_n^+}{\theta_n} \qquad \text{and} \qquad \phi_n^- = \frac{e^{-\lambda x-(-1)^r\lambda^{2r+1}t_r}Q_n^-}{\theta_n} \tag{5.6}$$

are solutions of the Schrödinger equation $(\mathcal{L}_n - E)\phi = 0$, for $r = 1, \dots, m$.

Next, consider the Schrödinger operators

$$\mathcal{L}_n^\pm = -\partial_{xx} + u_n^\pm$$

for the potentials $u_n^\pm$. In this section we compute fundamental matrices for the Schrödinger equations $\mathcal{L}_n^\pm\psi = 0$ and $(\mathcal{L}_n^\pm - E)\psi = 0$. Observe that in the first one apparently there is no dependence on E, but recall that the functions $u_n^\pm$ depend on λ, which is related to E by $\lambda^2 + E = 0$. So, both Schrödinger equations depend on E and hence, both fundamental matrices will depend on λ.

We have obtained the following result for $\mathcal{L}_n^\pm\psi = 0$:

Theorem 5.3. *Let n be a non negative integer. A fundamental matrix for the Schrödinger equation $\mathcal{L}_n^\pm\psi = (-\partial_{xx} + u_n^\pm)\psi = 0$ is:*

$$\mathcal{B}_{n,0}^\pm = \begin{pmatrix} \psi_{1,n}^\pm & \psi_{2,n}^\pm \\ \psi_{1,n,x}^\pm & \psi_{2,n,x}^\pm \end{pmatrix},$$

with

$$\psi_{1,n}^\pm(x, \tau_2^\pm, \dots, \tau_n^\pm, \lambda) = \frac{Q_{n-1}^\pm}{Q_n^\pm} \qquad \text{and} \qquad \psi_{2,n}^\pm(x, \tau_2^\pm, \dots, \tau_n^\pm, \lambda) = \frac{Q_{n+1}^\pm}{Q_n^\pm},$$

and the initial data for $n = 0$, $Q_{-1}^+ := -1$ and $Q_{-1}^- := 1$. Notice that $\psi_{2,n}^\pm = (\psi_{1,n+1}^\pm)^{-1}$.

Proof. We have that the functions $u_n^\pm$ are obtained from the Adler–Moser potentials u_n by performing Darboux–Crum transformations to them with the unadjusted functions ϕ_n^+ and ϕ_n^- defined in (5.6):

$$DT(\phi_n^\pm)u_n = u_n - 2(\log\phi_n^\pm)_{xx} = -2(\log Q_n^\pm)_{xx} = u_n^\pm. \tag{5.7}$$

Therefore, the Darboux–Crum transformations of the unadjusted solutions $\phi_{1,n}$ and $\phi_{2,n}$ defined in (5.5):

$$DT(\phi_n^\pm)\phi_{1,n} = \phi_{1,n,x} - \frac{\phi_{n,x}^\pm}{\phi_n^\pm}\phi_{1,n} = \mp\lambda^2\frac{Q_{n-1}^\pm}{Q_n^\pm} = \mp\lambda^2\psi_{1,n}^\pm,$$

$$DT(\phi_n^\pm)\phi_{2,n} = \phi_{2,n,x} - \frac{\phi_{n,x}^\pm}{\phi_n^\pm}\phi_{2,n} = \mp\frac{Q_{n+1}^\pm}{Q_n^\pm} = \mp\psi_{2,n}^\pm,$$

where we have used relations (4.40), (4.41) (4.42) and (4.43), are solutions of the same Schödinger equation as functions $\phi_{1,n}$ and $\phi_{2,n}$, but for these transformed potentials, i.e., they are solutions of the Schödinger equation $\mathcal{L}_n^{\pm}\psi = 0$. □

Remark 5.4. Since functions $\psi_{1,n}^{\pm}$ and $\psi_{2,n}^{\pm}$ are solutions of Schödinger equation $\mathcal{L}_n^{\pm}\psi = 0$, we have an analogous relation to (4.19) for polynomials $Q_n^{\pm}$:

$$Q_{n+1,xx}^{\pm}Q_n^{\pm} + Q_{n+1}^{\pm}Q_{n,xx}^{\pm} - 2Q_{n+1,x}^{\pm}Q_{n,x}^{\pm} = 0. \tag{5.8}$$

We can compute the determinant of this fundamental matrix in the same way we did for $\mathcal{B}_{n,0}^{(r)}$. In this case, we obtain:

Proposition 5.5. *We have:*

$$\det \mathcal{B}_{n,0}^{\pm} = 2n + 1.$$

Proof. It is immediate applying differential recursions (5.1) and (5.2). □

Using recursion formulas (5.1) and (5.2) we can compute these fundamental matrices for every n, analogously to who we have computed fundamental matrices $\mathcal{B}_{n,\lambda}^{(r)}$. We show the first cases in the following examples:

Example 5.6. We have the following fundamental solutions of $\mathcal{L}_n^{+}\psi = 0$ for potentials u_n^{+} computed in Example 5.1:

n	$\psi_{1,n}^{+}$	$\psi_{2,n}^{+}$
0	-1	$\lambda x - 1$
1	$\dfrac{1}{\lambda x - 1}$	$\dfrac{\lambda^2 x^3 - 3\lambda x^2 + 3x + \tau_2^{+}}{\lambda x - 1}$
2	$\dfrac{\lambda x - 1}{\lambda^2 x^3 - 3\lambda x^2 + 3x + \tau_2^{+}}$	$\dfrac{Q_3^{+}}{\lambda^2 x^3 - 3\lambda x^2 + 3x + \tau_2^{+}}$

where $Q_3^{+} = \lambda^3 x^6 - 6\lambda^2 x^5 + 15\lambda x^4 - 15x^3 + 5\lambda x^3\tau_2^{+} - 15x^2\tau_2^{+} - (\lambda\tau_3^{+} + 5(\tau_2^{+})^2)x + \tau_3^{+}$.

Example 5.7. We have the following fundamental solutions of $\mathcal{L}_n^{-}\psi = 0$ for potentials u_n^{-} computed in Example 5.1:

n	$\psi_{1,n}^{-}$	$\psi_{2,n}^{-}$
0	1	$\lambda x + 1$
1	$\dfrac{1}{\lambda x + 1}$	$\dfrac{\lambda^2 x^3 + 3\lambda x^2 + 3x + \tau_2^{-}}{\lambda x + 1}$
2	$\dfrac{\lambda x + 1}{\lambda^2 x^3 + 3\lambda x^2 + 3x + \tau_2^{-}}$	$\dfrac{Q_3^{-}}{\lambda^2 x^3 + 3\lambda x^2 + 3x + \tau_2^{-}}$

where $Q_3^{-} = \lambda^3 x^6 + 6\lambda^2 x^5 + 15\lambda x^4 + 15x^3 + 5\lambda x^3\tau_2^{-} + 15x^2\tau_2^{-} + (\lambda\tau_3^{-} + 5(\tau_2^{-})^2)x + \tau_3^{-}$.

Now, we show a fundamental matrix for the Schrödinger equation $(\mathcal{L}_n^{\pm} - E)\psi = 0$:

Theorem 5.8. *Let n be a non negative integer. A fundamental matrix for Schrödinger equation* $(\mathcal{L}_n^{\pm} - E)\psi = 0$ *is:*

$$\mathcal{B}_{n,\lambda}^{\pm} = \begin{pmatrix} \Delta_{1,n}^{\pm} & \Delta_{2,n}^{\pm} \\ \Delta_{1,n,x}^{\pm} & \Delta_{2,n,x}^{\pm} \end{pmatrix},$$

where

$$\Delta_{1,n}^{\pm}(x, t_r, \tau_2^{\pm}, \dots, \tau_n^{\pm}, \lambda) = \frac{1}{\phi_n^{\pm}} \quad \textit{and} \quad \Delta_{2,n}^{\pm}(x, t_r \tau_2^{\pm}, \dots, \tau_n^{\pm}, \lambda) = \frac{1}{\phi_n^{\pm}} \cdot \int (\phi_n^{\pm})^2 dx,$$

for functions $\phi_n^{\pm}(x, t_r, \tau_2^{\pm}, \dots, \tau_n^{\pm}, \lambda)$ *defined in* (5.6).

Proof. As in the proof of Theorem 5.3, we have that the potentials $u_n^{\pm}$ are obtained by Darboux–Crum transformations with the unadjusted functions $\phi_n^{\pm}$ defined by expressions (5.6):

$$DT(\phi_n^{\pm})u_n = u_n - 2(\log \phi_n^{\pm})_{xx} = -2(\log Q_n^{\pm})_{xx} = u_n^{\pm}.$$

Therefore, the Darboux–Crum transformations of the unadjusted solutions ϕ_n^+ and ϕ_n^-:

$$DT(\phi_n^+)\phi_n^- = \phi_{n,x}^- - \frac{\phi_{n,x}^+}{\phi_n^+}\phi_n^- = -\frac{2\lambda^{2n+1}}{\phi_n^+} = -2\lambda^{2n+1}\Delta_n^+, \tag{5.9}$$

$$DT(\phi_n^-)\phi_n^+ = \phi_{n,x}^+ - \frac{\phi_{n,x}^-}{\phi_n^-}\phi_n^+ = \frac{2\lambda^{2n+1}}{\phi_n^-} = 2\lambda^{2n+1}\Delta_n^-, \tag{5.10}$$

by Theorem 4.15, are solutions of the same Schödinger equation as functions ϕ_n^+ and ϕ_n^-, but for these transformed potentials, i.e., they are solutions of the Schödinger equations $(\mathcal{L}_n^+ - E)\psi = 0$ and $(\mathcal{L}_n^- - E)\psi = 0$ respectively.

Now, in order to find the second fundamental solution of each Schödinger equation, we use Remark 1.13 and we obtain:

$$\Delta_{2,n}^{\pm} = \Delta_{1,n}^{\pm} \int \frac{dx}{(\Delta_{1,n}^{\pm})^2} = \frac{1}{\phi_n^{\pm}} \cdot \int (\phi_n^{\pm})^2 dx.$$

This concludes the proof. ☐

We can also compute the determinant of $\mathcal{B}_{n,\lambda}^{\pm}$:

Proposition 5.9. *We have:*

$$\det \mathcal{B}_{n,\lambda}^{\pm} = 1.$$

Proof. It follows by direct computation for functions the $\Delta_{1,n}^{\pm}$ and $\Delta_{2,n}^{\pm}$ defined in Theorem 5.8. ☐

We show the first fundamental solutions of differential equations $(\mathcal{L}_n^{\pm} - E)\psi = 0$ in the next two examples.

Example 5.10. We have the following fundamental solutions of $(\mathcal{L}_n^+ - E)\Delta = 0$ for potentials u_n^+ computed in Example 5.1:

n	$\Delta_{1,n}^+$	$\Delta_{2,n}^+$
0	$e^{-\lambda x-(-1)^r\lambda^{2r+1}t_r}$	$e^{\lambda x+(-1)^r\lambda^{2r+1}t_r}$
1	$\dfrac{x\cdot e^{-\lambda x-(-1)^r\lambda^{2r+1}t_r}}{\lambda x-1}$	$\dfrac{e^{\lambda x+(-1)^r\lambda^{2r+1}t_r}(\lambda x-2)}{\lambda x-1}$
2	$\dfrac{e^{-\lambda x-(-1)^r\lambda^{2r+1}t_r}(x^3+\tau_2)}{\lambda^2x^3-3\lambda x^2+3x+\tau_2^+}$	$\Delta_{1,2}^+\displaystyle\int\frac{dx}{(\Delta_{1,2}^+)^2}$

Example 5.11. We have the following fundamental solutions of $(\mathcal{L}_n^- - E)\Delta = 0$ for potentials u_n^- computed in Example 5.1:

n	$\Delta_{1,n}^-$	$\Delta_{2,n}^-$
0	$e^{\lambda x+(-1)^r\lambda^{2r+1}t_r}$	$e^{-\lambda x-(-1)^r\lambda^{2r+1}t_r}$
1	$\dfrac{xe^{\lambda x+(-1)^r\lambda^{2r+1}t_r}}{\lambda x+1}$	$\dfrac{e^{-\lambda x-(-1)^r\lambda^{2r+1}t_r}(\lambda x+2)}{\lambda x+1}$
2	$\dfrac{e^{\lambda x+(-1)^r\lambda^{2r+1}t_r}(x^3+\tau_2)}{\lambda^2x^3+3\lambda x^2+3x+\tau_2^-}$	$\Delta_{1,2}^-\displaystyle\int\frac{dx}{(\Delta_{1,2}^-)^2}$

5.1.1 Darboux–Crum transformations and fundamental matrices

Let fix $n\in\mathbb{N}$ and consider the spectral problems $\mathcal{L}_n^\pm\psi = 0$ and $(\mathcal{L}_n^\pm - E)\Delta = 0$. Let $\mathcal{B}_{n,0}^\pm$ and $\mathcal{B}_{n,\lambda}^\pm$ be their corresponding fundamental matrices. We show next how we can obtain the corresponding fundamental matrices of the Schrödinger equations for levels $n+1$ and $n-1$ by means of Darboux–Crum transformations. For this, we will perform Darboux–Crum transformations by means of the functions $\psi_{1,n}^\pm$ and $\psi_{2,n}^\pm$.

Observe that whenever we perform the Darboux–Crum transformation with $\psi_{1,n}^\pm$, we obtain the formulas

$$DT(\psi_{1,n}^\pm)\psi_{2,n}^\pm = \psi_{2,n,x}^\pm - \frac{\psi_{1,n,x}^\pm}{\psi_{1,n}^\pm}\psi_{2,n}^\pm = c_{2,n-1}^\pm\frac{Q_n^\pm}{Q_{n-1}^\pm} = c_{2,n-1}^\pm\psi_{2,n-1}^\pm,$$

where $c_{2,n-1}^\pm\in\mathbf{C}$; and then it is a solution of the Schrödinger equation $\mathcal{L}_{n-1}^\pm\psi = 0$ for potential

$$DT(\psi_{1,n}^\pm)u_n^\pm = u_n^\pm - 2(\log\psi_{1,n}^\pm)_{xx} = -2(\log Q_{n-1}^\pm)_{xx} = u_{n-1}^\pm. \tag{5.11}$$

By an analogous argument to that of the proof of Theorem 4.7, we recover the solution $\psi_{1,n-1}^\pm$ from the transformed solution $\psi_{2,n-1}^\pm$ and we get the fundamental matrix $\mathcal{B}_{n-1,0}^\pm$.

Futhermore, the function

$$DT(\psi_{1,n}^\pm)\Delta_{1,n}^\pm = \Delta_{1,n,x}^\pm - \frac{\psi_{1,n,x}^\pm}{\psi_{1,n}^\pm}\Delta_{1,n}^\pm = c_{n-1}^\pm\frac{1}{\phi_{n-1}^\pm} = c_{n-1}^\pm\Delta_{1,n-1}^\pm,$$

where $c^{\pm}_{n-1} \in \mathbf{C}$, is a solution of the Schrödinger equation $(\mathcal{L}^{\pm}_{n-1} - E)\Delta = 0$ for potential (5.11). Using this solution, we define $\Delta^{\pm}_{2,n-1}$ as

$$\Delta^{\pm}_{2,n-1} := \Delta^{\pm}_{1,n-1} \int \frac{dx}{(\Delta^{\pm}_{1,n-1})^2}$$

and, in this way, we get the fundamental matrix $\mathcal{B}^{\pm}_{n-1,\lambda}$.

By a totally analogous process for the Darboux–Crum transformation with $\psi^{\pm}_{2,n}$ we get the fundamental matrices $\mathcal{B}^{\pm}_{n+1,0}$ and $\mathcal{B}^{\pm}_{n+1,\lambda}$ for the Schrödinger equations for potential:

$$DT(\psi^{\pm}_{2,n})u^{\pm}_n = u^{\pm}_n - 2(\log \psi^{\pm}_{2,n})_{xx} = -2(\log Q^{\pm}_{n+1})_{xx} = u^{\pm}_{n+1}. \tag{5.12}$$

So, the family of λ-parameterized potentials $u^{\pm}_n$ can be directly constructed by means of Darboux–Crum transformations starting from the potential for $n = 0$, i.e., $u^{\pm}_0 = 0$. This shows that these generalized potentials have a behaviour under Darboux–Crum transformations similar to that of the Adler–Moser potentials. In particular, this implies that we can move through the fundamental matrices of the different levels n of the Schrödinger equations for potential $u^{\pm}_n$ by means of Darboux–Crum transformations, in the same way as in Chapter 4.

5.2 Generalized KdV$_r$ potentials

Next we prove that, for fix n, there exists a unique choice of the constants (with respect to ∂_x) τ^+_i and τ^-_i for $i = 2, \ldots, n$, such that functions $u^{\pm}_n$ are solutions of the KdV$_r$ equation (3.51) for each r and constants $c_j = 0$, for $j = 1, \ldots, r$, i.e., they are KdV$_r$ potentials. To do that, we adapt the proof of Theorem 4.2 due to Adler and Moser for these functions. Before formulating our result, we need to introduce some technical notions and notations concerning Darboux–Crum transformations for these potentials that will be necessary in the proof.

First, we take the fundamental solutions $\psi^{\pm}_{1,n}$ and $\psi^{\pm}_{2,n}$ of Schrödinger equation $\mathcal{L}^{\pm}_n\psi = 0$ defined in Theorem 5.3. Next, we define the logarithmic derivatives of these functions by $\nu^{\pm}_{1,n} = (\log \psi^{\pm}_{1,n})_x$ and $\nu^{\pm}_{2,n} = (\log \psi^{\pm}_{2,n})_x$. As we have already shown in subsection 1.3.1, this functions are, by construction, solutions of the Riccati equation:

$$\nu_x = u^{\pm}_n - \nu^2, \tag{5.13}$$

which allows us to factor the Schrödinger operator $\mathcal{L}^{\pm}_n = -\partial_{xx} + u^{\pm}_n$ as

$$(-\partial_x - \nu)(\partial_x - \nu) = -\partial_{xx} + \nu^2 + \nu_x = -\partial_{xx} + u^{\pm}_{r,n} = \mathcal{L}^{\pm}_{r,n},$$

if and only if ν is a solution of the Riccati equation (5.13). In particular, setting $\nu = \nu^{\pm}_{1,n}$ or $\nu = \nu^{\pm}_{2,n}$ we get the factorization. Following the same procedure as in subsection 1.3.1, we exchange the factors in previous expression and we obtain:

$$(\partial_x - \nu)(-\partial_x - \nu) = -\partial_{xx} + \nu^2 - \nu_x.$$

Using differential relation (5.8) it is easy to verify that

$$u_{n+1}^{\pm} = (\nu_{2,n}^{\pm})^2 - \nu_{2,n,x}^{\pm} \qquad \text{and} \qquad u_{n-1}^{\pm} = (\nu_{1,n}^{\pm})^2 - \nu_{1,n,x}^{\pm}. \tag{5.14}$$

Hence, we get:

$$\begin{aligned}(\partial_x - \nu_{2,n}^{\pm})(-\partial_x - \nu_{2,n}^{\pm}) &= -\partial_{xx} + u_{n+1}^{\pm} = \mathcal{L}_{r,n+1}^{\pm},\\ (\partial_x - \nu_{1,n}^{\pm})(-\partial_x - \nu_{1,n}^{\pm}) &= -\partial_{xx} + u_{n-1}^{\pm} = \mathcal{L}_{r,n-1}^{\pm}.\end{aligned}$$

Therefore, by equations (5.13) and (5.14), we can conclude, as we already knew (see (5.11) and (5.12)), that

$$\begin{aligned}u_{n+1}^{\pm} &= u_n^{\pm} - 2\nu_{2,n,x}^{\pm} = DT(\psi_{2,n}^{\pm})u_n^{\pm},\\ u_{n-1}^{\pm} &= u_n^{\pm} - 2\nu_{1,n,x}^{\pm} = DT(\psi_{1,n}^{\pm})u_n^{\pm}.\end{aligned} \tag{5.15}$$

Futhermore, the functions $u_n^{\pm}$, $u_{n+1}^{\pm}$ and $\nu_{2,n,x}^{\pm}$ verify both statements of Proposition 3.24, by equation (5.15).

Finally, we can stablish the following result (the function g_r is defined by equation (3.57)):

Theorem 5.12. *There is a unique choice of rational functions* $\gamma_{rj}^{\pm} = \gamma_{rj}(\tau_2^{\pm}, \dots, \tau_j^{\pm})$ *and differential operators*

$$\Xi_r^{\pm} = \sum_{j=1}^{\infty} \gamma_{rj}^{\pm} \frac{\partial}{\partial \tau_j^{\pm}}$$

such that

$$2f_{r+1,x}(u_n^{\pm}) = \Xi_r u_n^{\pm} \tag{5.16}$$

for $n = 0, 1, 2, \dots,$ *and*

$$g_r(\nu_{2,n}^{\pm}) = \Xi_r^{\pm}\nu_{2,n}^{\pm} \quad \textit{where} \quad \nu_{2,n}^{\pm} = \frac{Q_{n+1,x}^{\pm}}{Q_{n+1}^{\pm}} - \frac{Q_{n,x}^{\pm}}{Q_n^{\pm}}, \tag{5.17}$$

since $u_n^{\pm}$ *and* $\nu_{2,n}^{\pm}$ *depend only on finitely many variables the sum breaks off. In other words, if the* $\tau_j^{\pm}$ *satisfy*

$$\frac{d\tau_j^{\pm}}{dt_r} = \gamma_{rj}(\tau_2^{\pm}, \dots, \tau_j^{\pm}), \quad j \leq n, \tag{5.18}$$

then $u_n^{\pm} = u_n^{\pm}(\tau_2^{\pm}, \dots, \tau_n^{\pm})$ *solves the equation* $u_{t_r} = 2f_{r+1,x}(u)$.

We would like to point out that the proof of this theorem is analogous to the one Adler and Moser gave in [7] page 18, but adapted to our potentials.

Proof. We prove it by induction on n. For $n = 0$ we have $Q_0^{\pm} = 1$, so $u_0^{\pm} = 0$. Thus, $2f_{r+1,x}(u_0^{\pm}) = 0$.

We suppose it true for n, i.e., we suppose that we have fixed the functions $\gamma_{rj}^{\pm} = \gamma_{rj}(\tau_2^{\pm}, \ldots, \tau_j^{\pm})$, for $j = 1, \ldots, n$ such that (5.16) holds. Now, we prove it for $n+1$. For this, we define

$$\Xi_r^0 = \Xi_r^{\pm}|_{\gamma_{r,n+1}^{\pm}=0}.$$

Since $u_n^{\pm}$ does not depend on $\tau_{r,n+1}^{\pm}$, by induction hypothesis we have that $2f_{r+1,x}(u_n^{\pm}) = \Xi_r^0 u_n^{\pm}$. By Proposition 3.24 (1) and equation (5.13) we have:

$$(2\nu_{2,n}^{\pm}+\partial_x)g_r(\nu_{2,n}^{\pm}) = 2f_{r+1,x}(u_n^{\pm}) = \Xi_r^0 u_n^{\pm} = \Xi_r^0((\nu_{2,n}^{\pm})^2+\nu_{2,n,x}^{\pm}) = (2\nu_{2,n}^{\pm}+\partial_x)\Xi_r^0\nu_{2,n}^{\pm}.$$

Thus,

$$(2\nu_{2,n}^{\pm} + \partial_x)(g_r(\nu_{2,n}^{\pm}) - \Xi_r^0\nu_{2,n}^{\pm}) = 0.$$

It is easy to prove that $(\psi_{2,n}^{\pm})^{-2} \in \ker(2\nu_{2,n}^{\pm} + \partial_x)$. So, we can set

$$g_r(\nu_{2,n}^{\pm}) - \Xi_r^0\nu_{2,n}^{\pm} = c(\psi_{2,n}^{\pm})^{-2}, \tag{5.19}$$

for $c = c(\tau_2^{\pm}, \ldots, \tau_{r,n+1}^{\pm})$.

Moreover, we have that:

$$\begin{aligned} g_r(\nu_{2,n}^{\pm}) = \Xi_r^{\pm}\nu_{2,n}^{\pm} = \sum_{j=1}^{\infty}\gamma_{rj}^{\pm}\frac{\partial}{\partial\tau_j^{\pm}}\nu_{2,n}^{\pm} &= \Xi_r^0\nu_{2,n}^{\pm} + \gamma_{r,n+1}^{\pm}\frac{\partial}{\partial\tau_{r,n+1}^{\pm}}\nu_{2,n}^{\pm} \\ &= g_r(\nu_{2,n}^{\pm}) - c(\psi_{2,n}^{\pm})^{-2} + \gamma_{r,n+1}\frac{\partial}{\partial\tau_{r,n+1}^{\pm}}\nu_{2,n}^{\pm}, \end{aligned}$$

by equation (5.19). Hence,

$$0 = \gamma_{r,n+1}^{\pm}\frac{\partial}{\partial\tau_{r,n+1}^{\pm}}\nu_{2,n}^{\pm} - c(\psi_{2,n}^{\pm})^{-2}. \tag{5.20}$$

There exists a unique choice for the coefficient $\gamma_{r,n+1}^{\pm} = \gamma_{r,n+1}(\tau_2^{\pm}, \ldots, \tau_{r,n+1}^{\pm})$ such that $c = 0$ if we replaced Ξ_r^0 by $\Xi_r^{\pm}$ in equation (5.19). For this, we have to compute $\partial_{\tau_{r,n+1}^{\pm}}\nu_{2,n}^{\pm}$. In order to compute it, we derivate expressions (5.1) and (5.2) with respect to $\tau_{r,n+1}^{\pm}$. As $Q_n^{\pm}$ and $Q_{n-1}^{\pm}$ do not depend on $\tau_{r,n+1}^{\pm}$, we obtain:

$$(Q_{n+1,x}^{\pm})_{\tau_{r,n+1}^{\pm}}Q_{n-1}^{\pm} - (Q_{n+1}^{\pm})_{\tau_{r,n+1}^{\pm}}Q_{n-1,x}^{\pm} = 0.$$

From this expression, we get

$$(Q_{n+1}^{\pm})_{\tau_{r,n+1}^{\pm}} = Q_{n-1}^{\pm}. \tag{5.21}$$

Then,

$$\frac{\partial}{\partial\tau_{r,n+1}^{\pm}}\nu_{2,n}^{\pm} = \frac{\partial}{\partial\tau_{r,n+1}^{\pm}}\left(\frac{Q_{n+1,x}^{\pm}}{Q_{n+1}^{\pm}} - \frac{Q_{n,x}^{\pm}}{Q_n^{\pm}}\right) = \left(\frac{Q_{n-1}^{\pm}}{Q_{n+1}^{\pm}}\right)_x = -(2n+1)(\psi_{2,n}^{\pm})^{-2},$$

by equations (5.1), (5.2) and (5.21). Replacing this equality in expression (5.20) yields to

$$0 = -(2n+1)\gamma^{\pm}_{r,n+1}(\psi^{\pm}_{2,n})^{-2} - c(\psi^{\pm}_{2,n})^{-2},$$

therefore, taking

$$\gamma^{\pm}_{r,n+1} = -\frac{c}{2n+1}$$

we obtain

$$g_r(\nu^{\pm}_{2,n}) = \Xi^{\pm}_r \nu^{\pm}_{2,n}.$$

Finally, from equations 3.24 (2) and (5.14) we can conclude that

$$\begin{aligned}2f_{r+1,x}(u^{\pm}_{n+1}) &= (2\nu^{\pm}_{2,n} - \partial_x)g_r(\nu^{\pm}_{2,n}) = (2\nu^{\pm}_{2,n} - \partial_x)\Xi^{\pm}_r\nu^{\pm}_{2,n} = \Xi^{\pm}_r((\nu^{\pm}_{2,n})^2 - \nu^{\pm}_{2,n,x})\\ &= \Xi^{\pm}_r u^{\pm}_{n+1}.\end{aligned}$$

And this ends the proof. □

We call these KdV potentials *generalized Adler–Moser potentials*, since they extend the ones of Adler and Moser in the sense we saw at the begining of the chapter.

Remark 5.13. As we already explained in Remark 4.3, Theorem 5.12 shows that for each level r the formulas (5.3) and (5.4) for $Q^{\pm}_n$ are solutions of the KdV$_r$ equation for some choice of functions $\tau^{\pm}_2, \dots, \tau^{\pm}_j \in \mathbf{C}(\lambda, t_r)$. Again, we write $\mathbf{C}(\lambda)$ to emphasize the dependence on λ. Hence these functions $\tau^{\pm}_2, \dots, \tau^{\pm}_j$ must be adapted in terms of t_r to get a solution of the KdV$_r$ equation by means of Theorem 5.12.

We will call *adjusted potentials* $u^{\pm}_{r,n}$ and *adjusted solutions* $\psi^{\pm}_{1,r,n}, \psi^{\pm}_{2,r,n}, \Delta^{\pm}_{1,r,n}$ and $\Delta^{\pm}_{2,r,n}$, whenever the functions $\tau^{\pm}_2, \dots, \tau^{\pm}_n \in \mathbf{C}(\lambda, t_r)$ are fixed by means of Theorem 5.12. Otherwise, will call *unadjusted potentials* $u^{\pm}_n$ and *unadjusted solutions* $\psi^{\pm}_{1,n}, \psi^{\pm}_{2,n}, \Delta^{\pm}_{1,n}$ and $\Delta^{\pm}_{2,n}$.

Remark 5.14. When we computed the adjustment of the $\tau^{\pm}_i$ for $Q^{\pm}_{r,n}$ in $\phi^{\pm}_{r,n}$ in Chapter 4, we do it in such a way that the matrix $\mathcal{B}^{(r)}_{n,\lambda}$ is a fundamental matrix of system (4.26) for potential $u_{r,n}$, i.e, $u_{r,n}$ is a solution of KdV$_r$ equation. This means that this adjustment is an adjustment for KdV$_r$. For this reason, the adjustment for the potentials $u^{\pm}_{r,n}$ will be the same one that we took for the functions $Q^{\pm}_{r,n}$ in Chapter 4. We will show this on an example to clarify the importance of the adjustments according to the level in the KdV hierarchy.

Example 5.15. As an example of adjusted potentials, we show the first of them for $n = 0, 1, 2$ and $r = 1$, with the explicit choice of functions $\tau^{\pm}_2, \dots, \tau^{\pm}_n$, i.e., these potentials are solution of the KdV$_1$ equation for $c_1 = 0$: $u_{t_1} = \frac{3}{2}uu_x - \frac{1}{4}u_{xxx}$. The computations were made using SAGE.

$u^{+}_{1,n}$	$u^{-}_{1,n}$	$(\tau^{\pm}_2, \dots, \tau^{\pm}_n)$
0	0	
$\frac{2\lambda^2}{(\lambda x-1)^2}$	$\frac{2\lambda^2}{(\lambda x+1)^2}$	
$\frac{6(\lambda x-1)(\lambda^3x^3-3\lambda^2x^2+3\lambda x-6\lambda^3t_1-3)}{(\lambda^2x^3-3\lambda x^2+3x+3\lambda^2t_1)^2}$	$\frac{6(\lambda x+1)(\lambda^3x^3+3\lambda^2x^2+3\lambda x-6\lambda^3t_1+3)}{(\lambda^2x^3+3\lambda x^2+3x+3\lambda^2t_1)^2}$	$(3\lambda^2t_1)$

5.2.1 Some explicitly adjusted examples

Here we present the computations of fundamental solutions of Schrödinger equation for adjusted potentials $u^{\pm}_{r,n}$ for $r = 1$, i.e., they are solutions of KdV$_1$ equation (for $c_1 = 0$):

$$u_{t_1} = \frac{3}{2}uu_x - \frac{1}{4}u_{xxx}.$$

In the examples we also show explicitly the adjustment of the functions $\tau^{\pm}_i$ in each case. Observe that the solutions of the system (5.18) are in general non linear in t_r. All the computations were made with SAGE.

1. We find the following fundamental solutions for Schrödinger equation $\mathcal{L}^{\pm}_{r,n}\psi = (-\partial_{xx} + u^{\pm}_{r,n})\psi = 0$ for same potentials as in Example 5.15, for $n = 0, 1$ and 2.

$$\begin{array}{ccc}
\psi^{+}_{1,1,n} & \psi^{+}_{2,1,n} & (\tau^{+}_2, \dots, \tau^{+}_n) \\
-1 & \lambda x - 1 & \\
\dfrac{1}{\lambda x - 1} & \dfrac{\lambda^2x^3 - 3\lambda x^2 + 3x + 3\lambda^2 t_1}{\lambda x - 1} & (3\lambda^2 t_1) \\
\dfrac{\lambda x - 1}{\lambda^2x^3 - 3\lambda x^2 + 3x + 3\lambda^2 t_1} & \dfrac{Q^{+}_3}{\lambda^2x^3 - 3\lambda x^2 + 3x + 3\lambda^2 t_1} & (3\lambda^2 t_1, -45(t_1 + \lambda^3 t_1^2))
\end{array}$$

where

$$\begin{aligned}Q^{+}_3(\lambda, x, t_1) =& \lambda^3x^6 - 6\lambda^2x^5 + 15\lambda x^4 - 15x^3 + \\ & 15\lambda^3x^3t_1 - 45\lambda^2x^2t_1 + 45\lambda x t_1 - 45\lambda^3t_1^2 - 45t_1.\end{aligned}$$

And also,

$$\begin{array}{ccc}
\psi^{-}_{1,1,n} & \psi^{-}_{2,1,n} & (\tau^{-}_2, \dots, \tau^{-}_n) \\
1 & \lambda x + 1 & \\
\dfrac{1}{\lambda x + 1} & \dfrac{\lambda^2x^3 + 3\lambda x^2 + 3x + 3\lambda^2 t_1}{\lambda x + 1} & (3\lambda^2 t_1) \\
\dfrac{\lambda x + 1}{\lambda^2x^3 + 3\lambda x^2 + 3x + 3\lambda^2 t_1} & \dfrac{Q^{-}_3}{\lambda^2x^3 + 3\lambda x^2 + 3x + 3\lambda^2 t_1} & (3\lambda^2 t_1, 45(t_1 - \lambda^3 t_1^2))
\end{array}$$

where

$$\begin{aligned}Q^{-}_3(\lambda, x, t_1) =& \lambda^3x^6 + 6\lambda^2x^5 + 15\lambda x^4 + 15x^3 + \\ & 15\lambda^3x^3t_1 + 45\lambda^2x^2t_1 + 45\lambda x t_1 - 45\lambda^3t_1^2 + 45t_1.\end{aligned}$$

2. For the Schrödinger equation $(\mathcal{L}^{\pm}_{r,n} - E)\Delta = (-\partial_{xx} + u^{\pm}_{r,n} - E)\Delta = 0$ we have the following fundamental solutions for same potentials as in Example 5.15, for

$n = 0, 1$ and 2. Since these functions also depend on the functions $\tau_2, \dots, \tau_n$, we specify the choice for them too.

$\Delta^+_{1,1,n}$	$\Delta^+_{2,1,n}$	$(\tau_2, \dots, \tau_n)$	$(\tau^+_2, \dots, \tau^+_n)$
$e^{-\lambda x+\lambda^3 t_1}$	$e^{\lambda x-\lambda^3 t_1}$		
$\dfrac{x \cdot e^{-\lambda x+\lambda^3 t_1}}{\lambda x - 1}$	$\dfrac{e^{\lambda x-\lambda^3 t_1}(\lambda x - 2)}{\lambda x - 1}$		
$\dfrac{e^{-\lambda x+\lambda^3 t_1}(x^3+3t_1)}{\lambda^2 x^3 - 3\lambda x^2 + 3x + 3\lambda^2 t_1}$	$\Delta^+_{1,1,2}\displaystyle\int \frac{dx}{(\Delta^+_{1,1,2})^2}$	$(3t_1)$	$(3\lambda^2 t_1)$

And also,

$\Delta^-_{1,1,n}$	$\Delta^-_{2,1,n}$	$(\tau_2, \dots, \tau_n)$	$(\tau^-_2, \dots, \tau^-_n)$
$e^{\lambda x-\lambda^3 t_1}$	$e^{-\lambda x+\lambda^3 t_1}$		
$\dfrac{x e^{\lambda x-\lambda^3 t_1}}{\lambda x + 1}$	$\dfrac{e^{-\lambda x+\lambda^3 t_1}(\lambda x + 2)}{\lambda x + 1}$		
$\dfrac{e^{\lambda x-\lambda^3 t_1}(x^3+3t_1)}{\lambda^2 x^3 + 3\lambda x^2 + 3x + 3\lambda^2 t_1}$	$\Delta^-_{1,1,2}\displaystyle\int \frac{dx}{(\Delta^-_{1,1,2})^2}$	$(3t_1)$	$(3\lambda^2 t_1)$

As we announced in Remark 5.14, this choice of the functions $\tau_2, \dots, \tau_n$ and $\tau^\pm_2, \dots, \tau^\pm_n$ is the same as in Section 4.4.2.

5.3 Factorization of Schrödinger operators for $E \neq 0$

Using the unadjusted solutions ϕ^+_n and ϕ^-_n of Schrödinger equation for $E \neq 0$ defined by (5.6) we can factor the Schrödinger operator

$$\mathcal{L}_n - E = -\partial_{xx} + u_n - E$$

as we showed in subsection 1.3.1. For this, we define the logarithmic derivatives $\sigma^+_n = (\log \phi^+_n)_x$ and $\sigma^-_n = (\log \phi^-_n)_x$. These functions are, by construction, solutions of the Riccati equation for $u = u_n$:

$$\sigma_x = u - E - \sigma^2. \tag{5.22}$$

Now, we replace σ by σ^+_n and σ^-_n in expression (1.22). This yields to two different factorizations of the Schrödinger operator $\mathcal{L}_n - E$:

$$(-\partial_x - \sigma^+_n)(\partial_x - \sigma^+_n) = -\partial_{xx} + (\sigma^+_n)^2 + \sigma^+_{n,x} = -\partial_{xx} + u_n - E = \mathcal{L}_n - E,$$
$$(-\partial_x - \sigma^-_n)(\partial_x - \sigma^-_n) = -\partial_{xx} + (\sigma^-_n)^2 + \sigma^-_{n,x} = -\partial_{xx} + u_n - E = \mathcal{L}_n - E,$$

by Riccati equation (5.22).

Now, we can obtain Riccati equations for potentials $u_n^{\pm}$. Using equations (5.7) and (5.22) we get:

$$\begin{aligned} u_n^+ &= u_n - 2(\log \phi_n^+)_{xx} = (\sigma_n^+)^2 + \sigma_{n,x}^+ + E - 2\sigma_{n,x}^+ = (\sigma_n^+)^2 - \sigma_{n,x}^+ + E, \\ u_n^- &= u_n - 2(\log \phi_n^-)_{xx} = (\sigma_n^-)^2 + \sigma_{n,x}^- + E - 2\sigma_{n,x}^- = (\sigma_n^-)^2 - \sigma_{n,x}^- + E. \end{aligned}$$

Using these Riccati equations we can exchange the factors in previous expressions to obtain, respectively, factorizations for the Schrödinger operators $\mathcal{L}_n^{\pm} - E$:

$$\begin{aligned} (\partial_x - \sigma_n^+)(-\partial_x - \sigma_n^+) &= -\partial_{xx} + (\sigma_n^+)^2 - \sigma_{n,x}^+ = -\partial_{xx} + u_n^+ - E = \mathcal{L}_n^+ - E, \\ (\partial_x - \sigma_n^-)(-\partial_x - \sigma_n^-) &= -\partial_{xx} + (\sigma_n^-)^2 - \sigma_{n,x}^- = -\partial_{xx} + u_n^- - E = \mathcal{L}_n^- - E. \end{aligned}$$

An interesting fact here is that if we denote by $\rho_{1,n}^{\pm} = (\log \Delta_{1,n}^{\pm})_x$, we get that $\rho_{1,n}^{\pm} = -\sigma_n^{\pm}$. So, we automatically obtain another factorizations of the Schrödinger operators in terms of these functions:

$$\begin{aligned} (-\partial_x - \rho_{1,n}^{\pm})(\partial_x - \rho_{1,n}^{\pm}) &= -\partial_{xx} + (\rho_{1,n}^{\pm})^2 + \rho_{1,n,x}^{\pm} = -\partial_{xx} + u_n^{\pm} - E = \mathcal{L}_n^{\pm} - E, \\ (\partial_x - \rho_{1,n}^{\pm})(-\partial_x - \rho_{1,n}^{\pm}) &= -\partial_{xx} + (\rho_{1,n}^{\pm})^2 - \rho_{1,n,x}^{\pm} = -\partial_{xx} + u_n - E = \mathcal{L}_n - E. \end{aligned}$$

Thus, we also get that

$$\begin{aligned} DT(\Delta_{1,n}^+)u_n^+ &= u_n^+ - 2(\log \Delta_{1,n}^+)_{xx} = u_n, \\ DT(\Delta_{1,n}^-)u_n^- &= u_n^- - 2(\log \Delta_{1,n}^-)_{xx} = u_n. \end{aligned}$$

From the above, we can see that $DT(\Delta_{1,n}^{\pm})$ are the opposite transformations to $DT(\phi_n^{\pm})$ and allow us to recover the Adler–Moser potentials. We can illustrate this behaviour of the Darboux–Crum transformations with the following diagram:

$$u_n \xrightarrow{DT(\phi_n^{\pm})} u_n^{\pm} \xrightarrow{DT(\Delta_{1,n}^{\pm})} u_n$$

5.4 Stationary case

We end this chapter analyzing the stationary potentials corresponding to $u_n^{\pm}$, i.e, we study the case $\tau_j^{\pm} = 0$, for all j, in $u_n^{\pm}$. Under this assumption, we will have the stationary polynomials $Q_n^{\pm(0)}(x,\lambda) = Q_{r,n}^{\pm}(x, t_r = 0, \lambda)$ and the stationary parameterized family of potentials $u_n^{\pm(0)}(x,\lambda) = u_{r,n}^{\pm}(x, t_r = 0, \lambda) = -2(\log Q_n^{\pm(0)})_{xx}$. We will prove that, for fix n, the potential $u_n^{\pm(0)}$ is a solution of the s-KdV$_n$ equation.

Applying recursions (5.1) and (5.2) we find for the first stationary polynomials:

n	$Q_n^{+(0)}$	$Q_n^{-(0)}$
0	1	1
1	$\lambda x - 1$	$\lambda x + 1$
2	$\lambda^2 x^3 - 3\lambda x^2 + 3x$	$\lambda^2 x^3 + 3\lambda x^2 + 3x$
3	$\lambda^3 x^6 - 6\lambda^2 x^5 + 15\lambda x^4 - 15x^3$	$\lambda^3 x^6 + 6\lambda^2 x^5 + 15\lambda x^4 + 15x^3$

and for the first stationary potentials:

n	$u_n^{+(0)}$	$u_n^{-(0)}$
0	0	0
1	$\dfrac{2\lambda^2}{(\lambda x-1)^2}$	$\dfrac{2\lambda^2}{(\lambda x+1)^2}$
2	$\dfrac{6(\lambda x-1)(\lambda^3x^3-3\lambda^2x^2+3\lambda x-3)}{x^2(\lambda^2x^2-3\lambda x+3)^2}$	$\dfrac{6(\lambda x+1)(\lambda^3x^3+3\lambda^2x^2+3\lambda x)}{x^2(\lambda^2x^2+3\lambda x+3)^2}$

The following result proves that the time independent polynomials $Q_n^{\pm(0)}$ can be generated recursively.

Lemma 5.16. *Functions $Q_n^{+(0)}$ and $Q_n^{-(0)}$ satisfy the following recursion formulas for $n \geq 2$:*

$$Q_0^{+(0)} = 1, \quad Q_1^{+(0)} = \lambda x - 1, \quad Q_n^{+(0)} = x^{n-1}(\lambda^2 x^n Q_{n-2}^{+(0)} - (2n-1)Q_{n-1}^{+(0)}), \tag{5.23}$$

$$Q_0^{-(0)} = 1, \quad Q_1^{-(0)} = \lambda x + 1, \quad Q_n^{-(0)} = x^{n-1}(\lambda^2 x^n Q_{n-2}^{-(0)} + (2n-1)Q_{n-1}^{-(0)}). \tag{5.24}$$

Proof. First we prove recursion (5.23). We prove it by induction on n. It holds for $Q_2^{+(0)} = \lambda^2x^3 - 3\lambda x^2 + 3x$. We suppose it true for n and prove it for $n+1$, i.e., we prove that $Q_{n+1}^{+(0)} = x^n(\lambda^2x^{n+1}Q_{n-1}^{+(0)} - (2n+1)Q_n^{+(0)})$. By equations (4.40) and (4.42) and Lemma 4.5, we know that:

$$\begin{aligned}
Q_{n+1}^{+(0)} &= \frac{\lambda Q_n^{+(0)}\theta_{n+1}^{(0)} + Q_{n,x}^{+(0)}\theta_{n+1}^{(0)} - Q_n^{+(0)}\theta_{n+1,x}^{(0)}}{\theta_n^{(0)}} \\
&= x^n\left(\lambda x Q_n^{+(0)} + xQ_{n,x}^{+(0)} - \frac{(n+1)(n+2)}{2}Q_n^{+(0)}\right), \qquad (5.25)\\
\lambda^2 Q_{n-1}^{+(0)} &= \frac{\lambda Q_n^{+(0)}\theta_{n-1}^{(0)} + Q_{n,x}^{+(0)}\theta_{n-1}^{(0)} - Q_n^{+(0)}\theta_{n-1,x}^{(0)}}{\theta_n^{(0)}} \\
&= \frac{\lambda x Q_n^{+(0)} + xQ_{n,x}^{+(0)} - \frac{n(n-1)}{2}Q_n^{+(0)}}{x^{n+1}}.
\end{aligned}$$

So, $\lambda^2 x^{n+1}Q_{n-1}^{+(0)} = \lambda x Q_n^{+(0)} + xQ_{n,x}^{+\ (0)} - \frac{n(n-1)}{2}Q_n^{+(0)}$. We replace this expression in the recursion formula for $Q_{n+1}^{+(0)}$ and we obtain:

$$\begin{aligned}
Q_{n+1}^{+(0)} &= x^n\left(\lambda^2x^{n+1}Q_{n-1}^{+(0)} - (2n+1)Q_n^{+(0)}\right) \\
&= x^n\left(\lambda x Q_n^{+(0)} + xQ_{n,x}^{+(0)} - \frac{n(n-1)}{2}Q_n^{+(0)} - (2n+1)Q_n^{+(0)}\right) \\
&= x^n\left(\lambda x Q_n^{+(0)} + xQ_{n,x}^{+(0)} - \frac{(n+1)(n+2)}{2}Q_n^{+(0)}\right).
\end{aligned}$$

So, this expression coincide with expression (5.25), which proves the statement.

Recursion formula (5.24) is proved in an analogous way. □

Using these recursions it is easy to verify inductively that polynomials $Q_n^{\pm(0)}$ are of the form:

$$\begin{aligned}
Q_n^{+(0)} &= x^{\frac{n(n-1)}{2}} \left(\lambda^n x^n + c_{n-1}^{+}\lambda^{n-1}x^{n-1} + \cdots + c_1^{+}\lambda x + c_0^{+}\right) \\
&= \lambda^n x^{\frac{n(n-1)}{2}} \prod_{i=1}^{n}(x - x_{n,i}^{+}) = \lambda^n x^{\frac{n(n-1)}{2}} R_n^{+(0)}, \qquad (5.26) \\
Q_n^{-(0)} &= x^{\frac{n(n-1)}{2}} \left(\lambda^n x^n + c_{n-1}^{-}\lambda^{n-1}x^{n-1} + \cdots + c_1^{-}\lambda x + c_0^{-}\right) \\
&= \lambda^n x^{\frac{n(n-1)}{2}} \prod_{i=1}^{n}(x - x_{n,i}^{-}) = \lambda^n x^{\frac{n(n-1)}{2}} R_n^{-(0)}, \qquad (5.27)
\end{aligned}$$

where $c_i^{\pm} \in \mathbf{C}$ do not depend on λ and $x_{n,i}^{\pm}$ belong to the algebraic closure of $\mathbf{C}(\lambda)$ for all i. Applying recursions (5.23) and (5.24) to these latter expressions we arrive to the following recursion formulas for polynomials $R_n^{\pm(0)} \in \mathbf{C}(\lambda)[x]$ for $n \geq 2$:

$$R_0^{+(0)} = 1\ , \qquad R_1^{+(0)} = x - \frac{1}{\lambda}\ , \qquad R_n^{+(0)} = x^2 R_{n-2}^{+(0)} - \frac{(2n-1)}{\lambda} R_{n-1}^{+(0)}, \qquad (5.28)$$

$$R_0^{-(0)} = 1\ , \qquad R_1^{-(0)} = x + \frac{1}{\lambda}\ , \qquad R_n^{-(0)} = x^2 R_{n-2}^{-(0)} + \frac{(2n-1)}{\lambda} R_{n-1}^{-(0)}. \qquad (5.29)$$

Finally, from expressions (5.26) and (5.27), we can compute the stationary potentials $u_n^{\pm(0)}$:

$$u_n^{+(0)} = -2(\log Q_n^{+(0)})_{xx} = \frac{n(n-1)}{x^2} + 2\sum_{i=1}^{n} \frac{1}{(x - x_i^{+})^2}, \qquad (5.30)$$

$$u_n^{-(0)} = -2(\log Q_n^{-(0)})_{xx} = \frac{n(n-1)}{x^2} + 2\sum_{i=1}^{n} \frac{1}{(x - x_i^{-})^2}. \qquad (5.31)$$

Now, take the Schrödinger operator for these potentials:

$$\mathcal{L}_n^{\pm(0)} = -\partial_{xx} + u_n^{\pm(0)}.$$

It is not difficult to prove that the fundamental matrices for the Schrödinger equations $\mathcal{L}_n^{\pm(0)}\psi = 0$ and $(\mathcal{L}_n^{\pm(0)} - E)\Delta = 0$ are the stationary matrices corresponding to $\mathcal{B}_{n,0}^{\pm}$ and $\mathcal{B}_{n,\lambda}^{\pm}$, i.e., they are, respectively, the matrices

$$\mathcal{B}_{n,0}^{\pm(0)} = \begin{pmatrix} \psi_{1,n}^{\pm(0)} & \psi_{2,n}^{\pm(0)} \\ \psi_{1,n,x}^{\pm(0)} & \psi_{2,n,x}^{\pm(0)} \end{pmatrix} \qquad \text{and} \qquad \mathcal{B}_{n,\lambda}^{\pm(0)} = \begin{pmatrix} \Delta_{1,n}^{\pm(0)} & \Delta_{2,n}^{\pm(0)} \\ \Delta_{1,n,x}^{\pm(0)} & \Delta_{2,n,x}^{\pm(0)} \end{pmatrix}$$

where

$$\begin{aligned}
\psi_{1,n}^{\pm(0)}(x,\lambda) &= \frac{Q_{n-1}^{\pm(0)}}{Q_n^{\pm(0)}}, \qquad & \psi_{2,n}^{\pm(0)}(x,\lambda) &= \frac{Q_{n+1}^{\pm(0)}}{Q_n^{\pm(0)}}, \\
\Delta_{1,n}^{\pm(0)}(x,\lambda) &= \frac{1}{\phi_n^{\pm(0)}}, \qquad & \Delta_{2,n}^{\pm(0)}(x,\lambda) &= \frac{1}{\phi_n^{\pm(0)}} \cdot \int (\phi_n^{\pm(0)})^2,
\end{aligned} \qquad (5.32)$$

for $\phi_n^{\pm(0)} = \frac{e^{\lambda x} Q_n^{\pm(0)}}{\theta_n^{(0)}}$.

Next, we prove that, for fix n, the potential $u_n^{\pm(0)}$ is a solution of the KdV_n equation. For this, we consider the following theorem due to Duistermaat and Grünbaum. It establishes six equivalent conditions that a function $u(x)$ must satisfy in order to be a s-KdV_n potential. For the reader's convenience, we reproduce it adapted to our notation:

Theorem 5.17 (Theorem 3.4 in [32])**.** *Let u be a rational function with $u(\infty) = 0$. Then the following properties are equivalent.*

1. *All eigenfunctions of $-\partial_x^2 + u(x)$ are meromorphic in $\mathbf{C}$.*
2. *All eigenfunctions of $-\partial_x^2 + u(x)$ are of the form $e^{\lambda x} a^+(x,\lambda) + e^{-\lambda x} a^-(x,\lambda)$ with $x \mapsto a^{\pm}(x,\lambda)$ rational and bounded at infinity.*
3. *At each pole p of u the Laurent expansion*
$$u(x) = \sum_{r \geq -2} c_r (x-p)^r$$
satisfies $c_{-2} = n_p(n_p+1)$, for some $n_p \in \mathbb{Z}_{>0}$, and $c_{2j-1} = 0$, for all integers j such that $0 \leq j \leq n_p$.
4. *u is obtained from $u = 0$ by finitely many rational Darboux–Crum transformations.*
5. *The potentials in the KdV–flow starting at u remain rational.*
6. *u is obtained from $\frac{n(n+1)}{x^2}$, with $n \in \mathbb{Z}_{\geq 0}$, by applying the flows in the KdV hierarchy.*

A function u which satisfies this Theorem is called *rational s-KdV_n potential.*

As a consequence of Theorem 5.17 we can finally prove the announced result:

Proposition 5.18. *Let $n \in \mathbb{Z}_{>0}$ be fixed, the function $u_n^{\pm(0)}$ is a solution of the s-KdV_n equation.*

Proof. First, we see that the functions $u_n^{\pm(0)}$ defined by (5.30) and (5.31) satisfy the hypothesis of the Theorem 5.17. Now, since the six statements of this theorem are equivalent, it suffices to prove one of them to conclude the result.

We prove by induction on n that statement 4 holds for our potentials $u_n^{\pm(0)}$. First, we know that $u_0^{\pm(0)} = 0$. Now, for fix n, we have that:

$$DT(\psi_{2,n}^{\pm(0)}) u_n^{\pm(0)} = u_n^{\pm(0)} - 2(\log \psi_{2,n}^{\pm(0)})_{xx} = -2(\log Q_{n+1}^{\pm(0)})_{xx} = u_{n+1}^{\pm(0)},$$

for $\psi_{2,n}^{\pm(0)}$ defined in (5.32). Hence, by induction, we can obtain potential $u_n^{\pm(0)}$ from $u_0^{\pm(0)} = 0$ by applying n Darboux–Crum transformations. So, the function $u_n^{\pm(0)}$ is a rational s-KdV_n potential. ☐

Remark 5.19. Another easy way to prove Proposition 5.18 is to prove that statement 1 holds for the fundamental solutions $\psi_{1,n}^{\pm\,(0)}$, $\psi_{2,n}^{\pm\,(0)}$, $\Delta_{1,n}^{\pm\,(0)}$ and $\Delta_{2,n}^{\pm\,(0)}$ defined in (5.32). It is immediate for functions $\psi_{1,n}^{\pm\,(0)}$, $\psi_{2,n}^{\pm\,(0)}$ and $\Delta_{1,n}^{\pm\,(0)}$ and it is not difficult to see that $\Delta_{2,n}^{\pm\,(0)}$ is meromorphic as well.

Chapter 6

Orthogonal differential systems and differential Galois Theory

In this chapter we construct Darboux transformations for orthogonal differential systems. The idea to do that is to extend the Darboux transformations for second order linear differential equations in matrix form to Darboux transformations for their second symmetric power differential equation, also in matrix form, and use the isomorphism between the Lie algebras $\mathfrak{sl}(2, \mathbf{C})$ and $\mathfrak{so}(3, \mathbf{C})$ to finally obtain Darboux transformations for orthogonal differential systems. We will give the explicit expressions of these Darboux transformations.

Along this chapter K will denote a differential field in the variable x with derivation $\partial_x = {}'$ and with field of constants $\mathbf{C}$ algebraically closed and of characteristic zero. Let $\lambda \in \mathbf{C}$ be a parameter.

The notations used in this chapter is different from the rest of the thesis. In particular, sometimes we will refer to second symmetric power systems as $\mathfrak{sym}^2(\mathfrak{sl}(2, K))$ systems and to orthogonal (differential) systems as $\mathfrak{so}(3, K)$ systems.

The work developed in this chapter is based on a joint work with Primitivo Acosta, Moulay Barkatou and Jacques-Arthur Weil.

6.1 Orthogonal differential systems

First we introduce orthogonal differential systems, or orthogonal systems for short (see [29], Book I, Chapter II). Hence, we consider the orthogonal system

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}' = \begin{pmatrix} 0 & h & -g \\ -h & 0 & f \\ g & -f & 0 \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \tag{6.1}$$

with the quadratic integral

$$\alpha^2 + \beta^2 + \gamma^2 = c, \tag{6.2}$$

with c a constant in $\mathbf{C}$. Darboux proved that the solutions of such systems depend on a Riccati equation. The result can be stated as follows.

Theorem 6.1 (Darboux, [29]). *Consider a differential system*

$$\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}' = \begin{pmatrix} 0 & h & -g \\ -h & 0 & f \\ g & -f & 0 \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \tag{6.3}$$

on the unit sphere $\alpha^2 + \beta^2 + \gamma^2 = 1$. Reparametrizing the sphere by

$$\alpha = \frac{1-uv}{u-v}, \qquad \beta = i\frac{1+uv}{u-v} \qquad \text{and} \qquad \gamma = \frac{u+v}{u-v}, \tag{6.4}$$

the system (6.3) is changed into the (Riccati) system

$$\left\{ \begin{array}{rcl} u' & = & \frac{1}{2}(g-if) - ihu + \frac{1}{2}(g+if)\,u^2, \\ v' & = & \frac{1}{2}(g-if) - ihv + \frac{1}{2}(g+if)\,v^2. \end{array} \right. \tag{6.5}$$

Furthermore,

$$u = \frac{\alpha + i\beta}{1-\gamma} = \frac{1+\gamma}{\alpha - i\beta} \qquad \text{and} \qquad v = -\frac{1-\gamma}{\alpha - i\beta} = -\frac{\alpha + i\beta}{1+\gamma}. \tag{6.6}$$

Remark 6.2. We can see that u and v correspond to the same Riccati equation, which for our purposes, is written in terms of the variable θ. Thus we have

$$\theta' = \omega + \mu\theta + \overline{\omega}\theta^2, \quad \omega = \frac{g-if}{2}, \quad \overline{\omega} = \frac{g+if}{2}, \quad \mu = -ih. \tag{6.7}$$

Recall that in Lemma 1.14 we saw that there exist a change of variable which allows us to move from a second order linear differential equation to a Riccati equation and vice versa. Thus, the connection between Riccati equations and orthogonal systems stated in this theorem has been essential to connect second order linear differential equations and orthogonal systems, as Darboux also showed in [29], Book I, Chapter II. In this work, we present an approach in which this relation is not neccesary and, in fact, appears as a consequence of our construction.

In [34, 35] Fedorov et. al. work with orthogonal systems (6.3) written in the form

$$Z' = Z \times \Omega, \quad Z = (\alpha, \beta, \gamma)^t, \quad \Omega = (f, g, h)^t, \quad f, g, h \in K, \tag{6.8}$$

to study the rigid solid. From now on, we will use the notation (6.8) for orthogonal systems instead of equation (6.3).

These systems include a interesting family of differential systems. In fact, following Darboux ([29], Book I, Chapter II), we consider the differential system:

$$\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}' = \begin{pmatrix} a & b & c\\ d & e & f\\ g & h & i\end{pmatrix} \cdot \begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}. \tag{6.9}$$

All the systems of this form that admit a quadratic integral of the form (6.2) can be reduced to

$$a = e = i = b + d = c + g = f + h = 0.$$

Hence, we get the orthogonal system

$$\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}' = \begin{pmatrix} 0 & b & c\\ -b & 0 & f\\ -c & -f & 0\end{pmatrix} \cdot \begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}.$$

Thus, the family of systems we are interested in is more general than the family of orthogonal systems. We will construct Darboux transformations for every system (6.9) which admits a quadratic first integral of the form (6.2).

6.2 Second order linear differential equations and Differential Galois Theory

As we explain at the begining, the idea to construct Darboux transformations for orthogonal systems is to extend the Darboux transformations for second order linear differential equations of the form:

$$\mathcal{P}_\lambda(y) = y'' + py' + (q - \lambda r)y = 0, \tag{6.10}$$

where $p, q, r \in K$ and $\lambda \in \mathbf{C}$ is a parameter.

Next theorem states the most general case of Darboux transformations for these differential equations.

Theorem 6.3 (Darboux, [26])**.** *Consider the differential equation*

$$y'' + py' + (q - \lambda r)y = 0, \tag{6.11}$$

where p, q, r are functions and λ is a parameter. Denote by y_λ a solution of (6.11) for λ. Suppose that we know a solution θ of system (6.11) for $\lambda = 0$. Let u_λ be a function defined by

$$u_\lambda = \frac{1}{\sqrt{r}}\left(y'_\lambda - \frac{\theta'}{\theta}y_\lambda\right) = \frac{1}{\sqrt{r}}\left(\frac{y'_\lambda}{y_\lambda} - \frac{\theta'}{\theta}\right)y_\lambda. \tag{6.12}$$

Then, for $\lambda \neq 0$, u_λ is a general solution of the new differential equation

$$u'' + pu' + (q_1 - \lambda r)u = 0, \quad \text{where } q_1 = \theta\sqrt{r}\left(\frac{p}{\theta\sqrt{r}} - \left(\frac{1}{\theta\sqrt{r}}\right)'\right)'. \tag{6.13}$$

The map (6.12) which transforms the family of equations (6.11) into the family (6.13) is called a Darboux transformation (DT).

Darboux also presented in [26, 28] the particular case for $r = 1$ and $p = 0$, which today is known as *Darboux transformation* (DT), but really is a corollary of the *general Darboux transformation* given in section 6.3.

We can see that the starting point in the philosophy of Darboux transformations is that there is a sort of covariance, which can be deduced from previous theorem. Observe that both equations have the same structure. Their only difference is that q is changed into q_1. In order to construct Darboux transformations for second symmetric power systems and orthogonal systems, we will have to preserve this covariance.

Now, we explain how Darboux transformations (DT) affect the Galois group of differential equation (6.10).

Let $(K, \partial) \subset (\widetilde{K}, \widetilde{\partial})$ be a differential ring's extension. Consider $K[\partial]$ and $\widetilde{K}[\partial]$ their corresponding rings of differential operators. Let $\varphi : K[\partial] \to \widetilde{K}[\partial]$ be a ring homomorphim with $\widetilde{K} = \varphi(K)$. We write $\widetilde{\mathcal{P}}$ for $\varphi(\mathcal{P})$ for any $\mathcal{P} \in K[\partial]$. Let G be the Galois group of $\mathcal{P}$, and $\widetilde{G}$ the corresponding of $\widetilde{\mathcal{P}}$. Moreover, L will denote the Picard–Vessiot extension corresponding to $\mathcal{P}$, and $\widetilde{L}$ the one for $\widetilde{\mathcal{P}}$.

The following definitions were introduced by P. Acosta-Humánez, J.J. Morales-Ruiz and J.-A. Weil in [2, 3, 4]:

1. φ is *isogaloisian* whether $G \simeq \widetilde{G}$,

2. φ is *strong isogaloisian* whether $\widetilde{L} = L$ and $\widetilde{K} = K$.

Let $G^0 \subset G$ be the identity connected component of G. We say that φ is a *virtually isogaloisian transformation* if $G^0 \simeq \widetilde{G}^0$ and it is a *virtually strong isogaloisian transformations* if $\widetilde{K}$ is an algebraic extension of K and $\widetilde{L}$ is an algebraic extension of L.

In [2, 3, 4], the authors proved the following results on the galoisian behaviour of Darboux transformations (DT):

Proposition 6.4 (Acosta-Humánez, Morales-Ruiz, Weil [2, 3, 4])**.** *The following statements hold.*

1. *DT is isogaloisian.*

2. *DT is strong isogaloisian when* $(\ln \theta)' \in K$*.*

3. *DT preserves the dimension of the eigenrings.*

Details and proofs of these results can be found in [4], see also [2, 3].

Without loss of generality, we assume from now on that there exists $w \in K$ such that $p = w'/w = (\log w)'$ in (6.10). Let L_λ be the Picard–Vesiot extension of $\mathcal{P}_\lambda(y) = 0$ for a fix λ and let G_λ be the Galois group of this differential extension. Then, G_λ is a subgroup of the group $SL(2, \mathbf{C})$ for every λ.

Next, we rewrite the differential equation $\mathcal{P}_\lambda(y) = 0$ in (6.10) as the following system of linear differential equations:

$$[A_\lambda]: \ X' = -(A_0 + \lambda N)X, \tag{6.14}$$

with

$$X = \begin{pmatrix} y \\ y' \end{pmatrix}, \quad A_0 = \begin{pmatrix} 0 & -1 \\ q & p \end{pmatrix}, \quad N = \begin{pmatrix} 0 & 0 \\ -r & 0 \end{pmatrix}.$$

Trivially we have that N^2 is the null matrix. Now, due to the Darboux transformation is covariant, after applying a Darboux transformation to this system we must obtain a differential system whose coefficients are all the same except from q, which in the transformed equation becomes q_1. So, we will have the new linear differential system

$$[\widetilde{A}_\lambda]: \widetilde{X}' = -(\widetilde{A}_0 + \lambda N)\widetilde{X}, \quad \widetilde{X} = \begin{pmatrix} u \\ u' \end{pmatrix}, \quad \widetilde{A}_0 = \begin{pmatrix} 0 & -1 \\ q_1 & p \end{pmatrix}, \tag{6.15}$$

where u and q_1 have the explicit form expressed in Theorem 6.3. We denote by $\widetilde{K} = K(q_1)$, the field of coefficients of system (6.15). Therefore, the equivalent result to Theorem 6.3 for matrix differential equations reads as follows.

Lemma 6.5. *Darboux transformation given in Theorem 6.3 is a virtually strong isogaloisian gauge transformation, where the gauge transformation matrix is given by*

$$P_\lambda := \frac{1}{\sqrt{r}} \begin{pmatrix} -z & 1 \\ \lambda r + z^2 + \frac{zr'}{2r} + z\frac{w'}{w} & -z - \frac{r'}{2r} - \frac{w'}{w} \end{pmatrix}, \tag{6.16}$$

where $z = (\log\theta)'$ and $p = (\log w)'$. Moreover, if $\sqrt{r} \in K$ and $z \in K$, this Darboux transformation is strong isogaloisian.

Proof. Consider the differential equations $\mathcal{P}_\lambda(y) = 0$ and $\widetilde{\mathcal{P}}_\lambda(u) = 0$ as in (6.10) and (6.13) respectively. We can assume $\{y_{1_\lambda}, y_{2_\lambda}\}$ and $\{u_{1_\lambda}, u_{2_\lambda}\}$ to be basis of solutions of $\mathcal{P}_\lambda(y) = 0$ and $\widetilde{\mathcal{P}}_\lambda(u) = 0$ respectively. Writing these equations in the form of the systems (6.14) and (6.15) respectively, we can see that

$$\mathrm{tr}(A_0 + \lambda N) = \mathrm{tr}(\widetilde{A}_0 + \lambda N) = p.$$

By Lemma 1.8 we have that

$$w_1'(y_{1_\lambda}, y_{2_\lambda}) + pw_1(y_{1_\lambda}, y_{2_\lambda}) = w_2(u_{1_\lambda}, u_{2_\lambda})' + pw_2(u_{1_\lambda}, u_{2_\lambda}) = 0,$$

where $w_1(y_{1_\lambda}, y_{2_\lambda})$ denotes the wronskian of $y_{1_\lambda}, y_{2_\lambda}$ and $w_2(u_{1_\lambda}, u_{2_\lambda})$ denotes the wronskian of $u_{1_\lambda}, u_{2_\lambda}$. Thus, we get

$$\frac{w_2'}{w_2} = \frac{w_1'}{w_1},$$

obtaining in this way that $w_2 = cw_1$, where c is a constant, which without loss of generality we can assume as $c = 1$.

Next, consider the matrix

$$P_\lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

By the change $\widetilde{X} = P_\lambda X$, we get that $u = ay + by'$ and $u' = a'y + ay' + b'y' + by''$, where $y'' = -py' - (q - \lambda r)y$. Thus, our matrix P_λ is given by

$$P_\lambda = \begin{pmatrix} a & b \\ a' - bq + br\lambda & a + b' - bp \end{pmatrix}.$$

Let $X_0 = (\theta, \theta')^t$ be a particular solution of the differential system $X_0' = A_0 X_0$ for $\lambda = 0$. Then one solution for a, b and q_1 is given by

$$a = -\frac{\theta'}{\theta\sqrt{r}}, \quad b = \frac{1}{\sqrt{r}}, \quad q_1 = -\theta\sqrt{r}\frac{d^2}{dx^2}\left(\frac{1}{\theta\sqrt{r}}\right) + \theta\sqrt{r}\frac{d}{dx}\left(\frac{p}{\theta\sqrt{r}}\right).$$

After replacing these expressions of a and b in the matrix P_λ, we arrive to the equation (6.16). We also see that $\det(P_\lambda) = -\lambda$ and $\det(P_0) = 0$.

Since $\sqrt{r}$ and $(\log\theta)'$ are algebraic functions over K, the differential field $\widetilde{K} = K(q_1)$ is an algebraic extension of the differential field K and the Picard–Vessiot extension $\widetilde{L}_\lambda$ of equation $\widetilde{\mathcal{P}}_\lambda(u) = 0$ is an algebraic extension of the Picard–Vessiot extension L_λ of $\mathcal{P}_\lambda(y) = 0$, thus, $G^0 = \widetilde{G}^0$.

Finally, if $\sqrt{r}$ and $(\log\theta)'$ belong to K, then $\widetilde{K} = K$ and $\widetilde{L}_\lambda = L_\lambda$, which implies that the Darboux transformation is strong isogaloisian and preserves the eigenrings. □

6.3 Second symmetric power systems and orthogonal systems

Now we explain the construction of linear differential systems associated to second symmetric power equations and orthogonal systems coming from second order linear differential equations of the form (6.10).

Consider the linear differential system $[A_0]$ as in equation (6.14) for $\lambda = 0$, where $p = (\ln w)'$, $w, q \in K$. We recall that its second symmetric power system is given by the linear differential system (see Subsection 1.2.3)

$$[\mathfrak{sym}^2(A_0)] : \; Y' = -S_2 Y, \tag{6.17}$$

for

$$Y_2 := \mathrm{Sym}^2(X) = \begin{pmatrix} y^2 \\ 2yy' \\ (y')^2 \end{pmatrix}, \quad S_2 := \mathfrak{sym}^2(A_0) = \begin{pmatrix} 0 & -1 & 0 \\ 2q & \frac{w'}{w} & -2 \\ 0 & q & 2\frac{w'}{w} \end{pmatrix}.$$

Given a fundamental matrix of equation (6.14)

$$X = \begin{pmatrix} y_1 & y_2 \\ y_1' & y_2' \end{pmatrix},$$

it is easy to check that a fundamental matrix for system (6.17) is

$$Y = \mathrm{Sym}^2(X) = \begin{pmatrix} y_1^2 & y_1y_2 & y_2^2 \\ 2y_1y_1' & y_1'y_2 + y_1y_2' & 2y_2y_2' \\ (y_1')^2 & y_1'y_2' & (y_2')^2 \end{pmatrix}. \tag{6.18}$$

Now, let $\sigma \in G$ be an automorphism of the Galois group of the equation $\mathcal{P}_0(y) = 0$ with matrix representation M_σ (recall that $G \subset SL(2, \mathbf{C})$), hence

$$M_\sigma = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \in G, \qquad \alpha_{11}\alpha_{22} - \alpha_{12}\alpha_{21} = 1,$$

in such a way that $\sigma(y_1) = \alpha_{11}y_1 + \alpha_{12}y_2$ and $\sigma(y_2) = \alpha_{21}y_1 + \alpha_{22}y_2$. Since $\{y_1^2, 2y_1y_2, y_2^2\}$ is a basis of solutions of $\mathfrak{sym}^2\mathcal{P}_0(y) = 0$, the differential automorphisms β of the Galois group of equation (6.17) satisfy

$$\beta(y_i^2) = (\beta(y_i))^2, \text{for } i = 1, 2,$$

and also $\beta(y_1y_2) = \beta(y_1)\beta(y_2)$. Thus, they must be also differential automorphisms of L_0, i.e., $\beta \in G$.

We denote by $\mathrm{Sym}^2(G)$ the Galois group of equation (6.17), then, given $\sigma \in \mathrm{Sym}^2(G)$, we have

$$\begin{aligned} \sigma(y_1^2) =& (\sigma(y_1))^2 = \alpha_{11}^2y_1^2 + 2\alpha_{11}\alpha_{12}y_1y_2 + \alpha_{12}^2y_2^2, \\ \sigma(y_2^2) =& (\sigma(y_2))^2 = \alpha_{21}^2y_1^2 + 2\alpha_{21}\alpha_{22}y_1y_2 + \alpha_{22}^2y_2^2, \\ \sigma(2y_1y_2) =& 2\sigma(y_1)\sigma(y_2) = 2\alpha_{11}\alpha_{21}y_1^2 + 2(\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21})y_1y_2 + 2\alpha_{12}\alpha_{22}y_2^2. \end{aligned}$$

Therefore,

$$\mathrm{Sym}^2(M_\sigma) = \begin{pmatrix} \alpha_{11}^2 & \alpha_{11}\alpha_{12} & \alpha_{12}^2 \\ 2\,\alpha_{11}\alpha_{21} & \alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21} & 2\,\alpha_{12}\alpha_{22} \\ \alpha_{21}^2 & \alpha_{21}\alpha_{22} & \alpha_{22}^2 \end{pmatrix} \in \mathrm{Sym}^2(G) \tag{6.19}$$

with determinant $\det \mathrm{Sym}^2(M_\sigma) = 1$. Hence, $\mathrm{Sym}^2(G)$ seen as group of matrices is a subgroup of $SL(3, \mathbf{C})$.

The procedure we just described allows us to compute explicitly a fundamental matrix and the Galois group of its second symmetric power system (6.17), whenever we know a fundamental matrix and the Galois group of differential system (6.14). Thus, we have proved the result:

Proposition 6.6. *Let X be a fundamental matrix of system* (6.14) *for $\lambda = 0$ and G be its Galois group. Then, $Y = \mathrm{Sym}^2(X)$ is a fundamental matrix for system* (6.17) *and* $\mathrm{Sym}^2(G)$ *is its Galois group.*

This situation can be illustrated by the following diagram:

$$\begin{tikzcd} {[A_0] : X' = -A_0 X} \arrow[r, rightsquigarrow] \arrow[d, "\mathfrak{sym}^2"] & G \arrow[d, "\mathrm{Sym}^2"] \\ {[\mathfrak{sym}^2(A_0)] : Y' = -S_2 Y} \arrow[r, rightsquigarrow] & \mathrm{Sym}^2(G) \end{tikzcd} \tag{6.20}$$

The next proposition, Proposition 6.7, provides us with a gauge transformation to go from the second symmetric power system (6.17) to an orthogonal system. This enables us to express equation $\mathcal{P}_0(y) = 0$ as a linear differential system in $\mathfrak{so}(3, K)$ written under the form $Z' = Z \times \Omega$ (equation (6.8)). As a consequence of this result, we can extend previous procedures for Galois groups to orthogonal systems, as it will be done next.

Proposition 6.7. *The gauge matrix Q given by*

$$Q = w \begin{pmatrix} 1 & 0 & -1 \\ i & 0 & i \\ 0 & -1 & 0 \end{pmatrix} \tag{6.21}$$

transforms the linear differential system (6.17) *to the orthogonal system $Z' = Z \times \Omega$, where $\Omega = (f, g, h)^t$, with $f = i(q-1)$, $g = q+1$ and $h = -ip$.*

Proof. Let $Y = (y^2, 2yy', y'^2)^t$ be a column solution of equation (6.17). Consider the gauge transformation $Z = QY$. Then Z will be a column solution of equation $Z' = (Q'Q^{-1} - QS_2Q^{-1})Z$, whenever $w \neq 0$. Since we have

$$Q'Q^{-1} - QS_2Q^{-1} = \begin{pmatrix} 0 & -ip & -(q+1) \\ ip & 0 & i(q-1) \\ q+1 & -i(q-1) & 0 \end{pmatrix} = \begin{pmatrix} 0 & h & -g \\ -h & 0 & f \\ g & -f & 0 \end{pmatrix},$$

we conclude that $Z' = Z \times \Omega$, as announced. $\square$

Remark 6.8. D. Blázquez-Sanz and J.J. Morales-Ruiz in [14], see also [13], have also given such a classical isomorphism between $\mathfrak{so}(3, \mathbf{C})$ and $\mathfrak{sl}(2, \mathbf{C})$, see Proposition 6.7 of [14]:

$$\begin{pmatrix} & 1 & \\ -1 & & \\ & & 0 \end{pmatrix} \mapsto \begin{pmatrix} \frac{i}{2} & 0 \\ 0 & -\frac{i}{2} \end{pmatrix}, \quad \begin{pmatrix} & & 1 \\ & 0 & \\ -1 & & \end{pmatrix} \mapsto \begin{pmatrix} 0 & \frac{1}{2} \\ -\frac{1}{2} & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & & \\ & & 1 \\ & -1 & \end{pmatrix} \mapsto \begin{pmatrix} 0 & -\frac{i}{2} \\ -\frac{i}{2} & 0 \end{pmatrix}.$$

As said before, it is well known that using Theorem 6.1 we can arrive from an orthogonal system to a second order linear differential equation (see Remark 6.2). For that it is enough to transform equation (6.7) into a second order linear differential equation, as shown in Lemma 1.14. We describe explicitly this situation in the following corollaries. The first of them gives us a dictionary between second order linear differential equations and orthogonal systems.

Corollary 6.9. *Let $f = i(q-1)$, $g = q+1$ and $h = -ip$. We consider the $\mathfrak{so}(3,K)$ system $Z' = Z \times \Omega$, for Ω as in (6.8). We introduce the second order linear differential equation $\mathcal{P}_0(y) := y'' + py' + qy = 0$. Note that*

$$\mathcal{P}_0(y) = y'' + \left(ih - \frac{g' + if'}{g + if}\right) y' + \frac{g^2 + f^2}{4} y. \tag{6.22}$$

1. *One solution of $Z' = Z \times \Omega$ is $(\alpha, \beta, \gamma)^t$ with*

$$\begin{aligned} \alpha &= \frac{4y_1'y_2' - (g+if)^2 y_1 y_2}{2(g+if)W(y_1,y_2)}, \quad \beta = -i\frac{4y_1'y_2' + (g+if)^2 y_1 y_2}{2(g+if)W(y_1,y_2)}, \\ \gamma &= \frac{y_1 y_2' + y_1' y_2}{W(y_1,y_2)}, \end{aligned} \tag{6.23}$$

   where $\{y_1, y_2\}$ is a basis of solutions of (6.22) and $W(y_1,y_2) = y_1 y_2' - y_1' y_2$.

2. *A basis of solutions for equation (6.22) is*

$$y_1 = e^{-\frac{1}{2}\int (g+if)\left(\frac{\alpha+i\beta}{1-\gamma}\right)dx}, \qquad y_2 = e^{\frac{1}{2}\int (g+if)\left(\frac{\alpha+i\beta}{1+\gamma}\right)dx},$$

   where $(\alpha, \beta, \gamma)^t$ is a solution of $Z' = Z \times \Omega$.

This explicit relation between the solutions of (6.22) and $Z' = Z \times \Omega$ was already showed by Darboux ([29], Book I, Chapter II).

Proof. By combination of Lemma 1.14 and Theorem 6.1 we obtain a second order linear differential equation from system (6.8) as follows (see also [34]). Consider the system given in equation (6.8), which, by Theorem 6.1, leads to the Riccati system (6.5) by means of the change of variables given in equation (6.4). Following Lemma 1.14, we consider the change of variable

$$\theta = -\frac{1}{\varpi}\frac{y'}{y}$$

to transform the Riccati equation (6.7) into the differential equation

$$y'' - \left(\mu + \frac{\varpi'}{\varpi}\right) y' + \omega \cdot \varpi \cdot y = 0, \quad \mu = -ih, \quad \omega = \frac{g - if}{2}. \tag{6.24}$$

In this way, if $(\alpha, \beta, \gamma)^t$ is one solution of system (6.8), then using equation (6.6) we obtain a basis of solutions of equation (6.22) given by

$$y_1 = e^{-\frac{1}{2}\int (g+if)\left(\frac{\alpha+i\beta}{1-\gamma}\right)dx}, \quad y_2 = e^{\frac{1}{2}\int (g+if)\left(\frac{\alpha+i\beta}{1+\gamma}\right)dx}.$$

On the other hand, if $\{y_1, y_2\}$ is a basis of solutions of equation (6.22), using (6.4) we arrive to a solution of equation (6.8) given by $(\alpha_2, \beta_2, \gamma_2)^t$ for

$$\alpha = \frac{4y_1'y_2' - (g+if)^2 y_1 y_2}{2(g+if)W(y_1,y_2)}, \quad \beta = -i\frac{4y_1'y_2' + (g+if)^2 y_1 y_2}{2(g+if)W(y_1,y_2)}, \quad \gamma = \frac{y_1 y_2' + y_1' y_2}{W(y_1,y_2)},$$

where $W(y_1, y_2) = y_1 y_2' - y_1' y_2$. □

Corollary 6.10. *A fundamental matrix for system $Z' = Z \times \Omega$ is*

$$Z := QY = w \begin{pmatrix} y_1^2 - (y_1')^2 & y_1y_2 - y_1'y_2' & y_2^2 - (y_2')^2 \\ i(y_1^2 + (y_1')^2) & i(y_1y_2 + y_1'y_2') & i(y_2^2 + (y_2')^2) \\ -2\,y_1y_1' & -y_1y_2' - y_1'y_2 & -2\,y_2y_2' \end{pmatrix}, \tag{6.25}$$

for Q and Y defined by (6.21) *and* (6.18) *respectively.*

This result extends Corollary 6.9, since it provides a fundamental matrix for the $\mathfrak{so}(3, K)$ system only in terms of a fundamental matrix for the differential system (6.14), without considering the Riccati equation. In fact, it is easy to verify that column solution (6.23) corresponds to the second column solution of fundamental matrix Z. Indeed, after substitution $f = i(q-1)$, $g = q+1$ and $h = -ip$ in (6.23) we get:

$$\alpha = \frac{-1}{W(y_1, y_2)}\left(y_1y_2 - y_1'y_2'\right), \quad \beta = \frac{-i}{W(y_1, y_2)}\left(y_1y_2 + y_1'y_2'\right), \quad \gamma = \frac{y_1y_2' + y_1'y_2}{W(y_1, y_2)}.$$

As $W'(y_1, y_2) = -pW(y_1, y_2)$, with $p = w'/w$, we have that $W(y_1, y_2) = w^{-1}$, which yields to the result.

Finally, we can compute the Galois group of system (6.8).

Corollary 6.11. *Using the fundamental matrix* (6.25)*, the matrices in the Galois group of the $\mathfrak{so}(3, K)$ system $Z' = Z \times \Omega$ are the matrices* $\mathrm{Sym}^2(M_\sigma)$ *of Proposition 6.6 (see formula* (6.19)*).*

Proof. Let σ be in the Galois group. It acts on Y via $\sigma(Y) = Y \cdot \mathrm{Sym}^2(M_\sigma)$ (see Proposition 6.6). Now, $\sigma(Q) = Q$ because $w \in Q$. So, we have $\sigma(Z) = \sigma(QY) = Q \cdot Y \cdot \mathrm{Sym}^2(M_\sigma) = Z \cdot \mathrm{Sym}^2(M_\sigma)$. □

Now, we can complete diagram (6.20) by adding the action on the $\mathfrak{so}(3, K)$ systems:

$$\begin{tikzcd} {[A_0] : X' = -A_0X} \arrow[r, squiggly] \arrow[d, "\mathfrak{sym}^2"] & G \arrow[d, "\mathrm{Sym}^2"] \\ {\mathfrak{sym}^2([A_0]) : Y' = -S_2Y} \arrow[r, squiggly] \arrow[d, "Q"] & \mathrm{Sym}^2(G) \arrow[d, equal] \\ {[\Omega] : Z' = -\Omega Z} \arrow[r, squiggly] & \mathrm{Sym}^2(G) \end{tikzcd} \tag{6.26}$$

The last result of this section links the first integrals (6.2) of orthogonal systems with the first integral of systems $\mathfrak{sym}^2([A_0])$.

Proposition 6.12. *Consider the systems $[\Omega] : Z' = \Omega Z$ with $Z = (\alpha, \beta, \gamma)^t$, and $[\mathfrak{sym}^2(A_0)] : Y' = \mathfrak{sym}^2(A_0)Y$ with $Y = (y_1, y_2, y_3)^t$. The system $[\Omega]$ admits the first integral $\alpha^2 + \beta^2 + \gamma^2$ and $[\mathfrak{sym}^2(A_0)]$ admits the first integral $w^2(4y_1y_3 - y_2^2)$.*

Proof. The first part is well known and is proved by Darboux in [29], see Chap. I, page 8 and Chap. II page 28.

The second part follows from the application of the gauge transformation of Proposition 6.7; in fact, it transforms the first integral $\alpha^2+\beta^2+\gamma^2$ of $[\Omega]$ into $w^2(4y_1y_3-y_2^2)$. But, this is still a constant of motion and hence a first integral of $[\mathfrak{sym}^2(A_0)]$. $\square$

6.4 DT for second symmetric power systems and orthogonal systems

The aim of this section is to construct Darboux transformations for diagram (6.26). By Proposition 6.4, we know that these Darboux transformations will preserve the Galois groups and the eigenrings of each equation.

First we will establish the behaviour and notations of the interaction of gauge transformations for linear differential systems and their m-th symmetric powers. For that, take a linear differential system

$$[A]: \; X' = -AX.$$

Let P be a gauge transformation for system $[A]$. The transformed system by P is

$$[P[A]]: \; \widetilde{X}' = -P[A]\widetilde{X},$$

with $\widetilde{X} = PX$ and $P[A] = PAP^{-1} - P'P^{-1}$. Now, the m-th symmetric power of $[A]$ is

$$[\mathfrak{sym}^m(A)]: \; \mathrm{Sym}^m(X)' = -\mathfrak{sym}^m(A)\cdot \mathrm{Sym}^m(X).$$

Then, the m-th symmetric power of the transformed system is

$$[\mathfrak{sym}^m(P[A])]: \; \mathrm{Sym}^m(\widetilde{X})' = -\mathfrak{sym}^m(P[A])\cdot \mathrm{Sym}^m(\widetilde{X}),$$

where $\mathrm{Sym}^m(\widetilde{X}) = \mathrm{Sym}^m(P)\cdot \mathrm{Sym}^m(X)$ and $\mathfrak{sym}^m(P[A]) = \mathrm{Sym}^m(P)\cdot[\mathfrak{sym}^m(A)]$.

Now, in order to construct the Darboux transformations for the second symmetric power system, coming from the general second order linear differential equation $\mathcal{P}_\lambda(y) = 0$ defined by (6.10), we extend Lemma 6.5. We present two ways to extend it. Like in previous section, we will obtain as corollaries the Darboux transformations for the $\mathfrak{so}(3,K)$ system.

For the first one, consider the linear differential system $[A_\lambda] : X' = -(A_0+\lambda N)X$ given by equation (6.14). Its second symmetric power system is given by the linear differential system

$$[\mathfrak{sym}^2(A_\lambda)] : Y' = -(S_2+\lambda N_2)Y, \tag{6.27}$$

where $Y = \mathrm{Sym}^2(X) = (y^2, 2yy', (y')^2)^t$ and $S_2+\lambda N_2 = \mathfrak{sym}^2(A_0+\lambda N)$ are given by

$$S_2 := \begin{pmatrix} 0 & -1 & 0 \\ 2q & \frac{w'}{w} & -2 \\ 0 & q & 2\frac{w'}{w} \end{pmatrix} \quad \text{and} \quad N_2 := \begin{pmatrix} 0 & 0 & 0 \\ -2r & 0 & 0 \\ 0 & -r & 0 \end{pmatrix}.$$

Recall that after applying the gauge transformation (6.16), system (6.14) is transformed into the linear differential system $[\widetilde{A}_\lambda] : \widetilde{X}' = -(\widetilde{A}_0 + \lambda N)\widetilde{X}$, defined by equation (6.15), whose second symmetric power system is given by the linear differential system

$$[\mathfrak{sym}^2(\widetilde{A}_\lambda)] : \widetilde{Y}' = -(\widetilde{S}_2 + \lambda N_2)\widetilde{Y}, \tag{6.28}$$

where $\widetilde{Y} = \mathrm{Sym}^2(\widetilde{X}) = (u^2, 2uu', (u')^2)^t$ and $\widetilde{S}_2 + \lambda N_2 = \mathfrak{sym}^2(\widetilde{A}_0 + \lambda N)$, for

$$\widetilde{S}_2 := \begin{pmatrix} 0 & -1 & 0 \\ 2q_1 & \frac{w'}{w} & -2 \\ 0 & q_1 & 2\frac{w'}{w} \end{pmatrix},$$

where u and q_1 have the explicit form expressed in Theorem 6.3. Thus, since $\widetilde{Y} = \mathrm{Sym}^2(\widetilde{X}) = \mathrm{Sym}^2(P_\lambda) \cdot \mathrm{Sym}^2(X) = \mathrm{Sym}^2(P_\lambda) \cdot Y$, the gauge transformation (6.16) also induces a transformation in the second symmetric power systems which sends system (6.27) to system (6.28). The following result establishes this idea.

Theorem 6.13 (First DT for $\mathfrak{sym}^2(\mathfrak{sl}(2, K))$)**.** *Consider the systems* $[\mathfrak{sym}^2(A_\lambda)]$: $Y' = -(S_2+\lambda N_2)Y$ *and* $[\mathfrak{sym}^2(\widetilde{A}_\lambda)] : \widetilde{Y}' = -(\widetilde{S}_2+\lambda N_2)\widetilde{Y}$ *given by* (6.27) *and* (6.28) *respectively. Let* $P_{1,\lambda}$ *be the matrix*

$$P_{1,\lambda} = \mathrm{Sym}^2(P_\lambda) = \frac{1}{r}\begin{pmatrix} z^2 & -z & 1 \\ 2\,z\,(z\tau - \lambda r) & \lambda r - 2\,z\tau & 2\,\tau \\ (z\tau - \lambda r)^2 & -\tau\,(z\tau - \lambda r) & \tau^2 \end{pmatrix}, \tag{6.29}$$

where P_λ *is defined by expression* (6.16)*,* $z = \frac{\theta'}{\theta}$ *and* $\tau = -z - \frac{w'}{w} - \frac{r'}{2r}$.

Then, $P_{1,\lambda}$ *is a Darboux transformation, i.e., a virtually strong isogaloisian gauge transformation, which sends system* $\mathfrak{sym}^2([A_\lambda])$ *to system* $\mathfrak{sym}^2([\widetilde{A}_\lambda])$.

Moreover, when $z \in K$*, this Darboux transformation is strong isogaloisian.*

Proof. Since matrix $P_{1,\lambda}$ is the second symmetric power matrix of matrix P_λ, defined by (6.16), given Y and $\widetilde{Y}$ solutions of equations (6.27) and (6.28) respectively, it satisfies $\widetilde{Y} = \mathrm{Sym}^2(P_\lambda) \cdot Y$. Therefore, we straightforwardly obtain the gauge transformation for the coefficient matrix

$$-(\widetilde{S}_2 + \lambda N_2) = P'_{1,\lambda} \cdot P^{-1}_{1,\lambda} - P_{1,\lambda} \cdot (S_2 + \lambda N_2) \cdot P^{-1}_{1,\lambda}.$$

We can trivially see that the coefficients of both systems only differs in q and q_1, in complete agreement with the philosophy of original Darboux transformation.

The rest of the theorem is proved following the same argument as in Proposition 6.5. □

There is another possibility of performing Darboux transformations to second symmetric power systems, that is, transform first system $[A_\lambda] : X' = -(A_0 + \lambda N)X$, given by equation (6.14), into a system in $\mathfrak{sl}(2, K)$ and perform all the previous process with the resulting system. By doing that, we ensure that the Galois group of equation (6.14) lies in $SL(2,\mathbf{C})$. In order to obtain an $\mathfrak{sl}(2,K)$ system, we let $X_1 := RX$, for

$$R := \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{w} \end{pmatrix}, \tag{6.30}$$

so that, the resulting system is the $\mathfrak{sl}(2,K)$ system

$$[B_\lambda] : X_1' = -(B_0 + \lambda N_1)X_1, \tag{6.31}$$

with $B_0 + \lambda N_1 \in \mathfrak{sl}(2,K)$, given by

$$X_1 = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \quad B_0 := \begin{pmatrix} 0 & -\frac{1}{w} \\ wq & 0 \end{pmatrix} \quad \text{and} \quad N_1 := \begin{pmatrix} 0 & 0 \\ -wr & 0 \end{pmatrix},$$

since R does not depend on λ. It is obvious that the Darboux transformation P_λ for system (6.14), given by (6.16), induces the Darboux transformation $RP_\lambda R^{-1}$ for system (6.31).

Now, we consider the second symmetric power system of system (6.31). This system is given by

$$[\mathfrak{sym}^2(B_\lambda)] : Y_1' = -(\widehat{S}_2 + \lambda \widehat{N}_2)Y_1, \tag{6.32}$$

where $Y_1 := \mathrm{Sym}^2(X_1) = (y_1^2, 2y_1y_2, y_2^2)^t$ and $\widehat{S}_2+\lambda\widehat{N}_2 = \mathfrak{sym}^2(B_0+\lambda N_1) \in \mathfrak{sl}(3,K)$, for

$$\widehat{S}_2 := \begin{pmatrix} 0 & -\frac{1}{w} & 0 \\ 2\,wq & 0 & \frac{-2}{w} \\ 0 & wq & 0 \end{pmatrix} \quad \text{and} \quad \widehat{N}_2 := \begin{pmatrix} 0 & 0 & 0 \\ -2\,wr & 0 & 0 \\ 0 & -wr & 0 \end{pmatrix}.$$

And we find the second expression for the Darboux transformation of second symmetric power systems:

Theorem 6.14 (Second DT for $\mathfrak{sym}^2(\mathfrak{sl}(2,K))$)**.** *Consider the system* $[\mathfrak{sym}^2(B_\lambda)]$: $Y_1' = -(\widehat{S}_2 + \lambda\widehat{N}_2)Y_1$ *given by* (6.32). *Let* $P_{2,\lambda}$ *be the matrix*

$$P_{2,\lambda} = \mathrm{Sym}^2(RP_\lambda R^{-1}) = \frac{1}{r}\begin{pmatrix} z^2 & \frac{-2zw}{1+w} & w \\ -\frac{(1+w)(\lambda r - z\tau)z}{w} & \lambda r - 2\,z\tau & (1+w)\,\tau \\ \frac{(r\lambda - z\tau)^2}{w} & \frac{2\tau\,(\lambda r - z\tau)}{1+w} & \tau^2 \end{pmatrix}, \tag{6.33}$$

where matrices P_λ *and* R *are defined by* (6.16) *and* (6.30) *respectively,* $z = \frac{\theta'}{\theta}$ *and* $\tau = -z - \frac{w'}{w} - \frac{r'}{2r}$.

Then, $P_{2,\lambda}$ *is a Darboux transformation for system* $[\mathfrak{sym}^2(B_\lambda)]$, *i.e., a virtually strong isogaloisian gauge transformation.*

Moreover, when $z \in K$, *this Darboux transformation is strong isogaloisian.*

Proof. As we have seen, a Darboux transformation for the $\mathfrak{sl}(2,K)$ system (6.31) is given by $\widetilde{X}_1 = RP_\lambda R^{-1}X_1$, where R and P_λ are defined by (6.30) and (6.16) respectively. The transformed system is the $\mathfrak{sl}(2,K)$ system

$$[\widetilde{B}_\lambda] := \widetilde{X}_1' = -(\widetilde{B}_0 + \lambda N_1)\widetilde{X}_1, \tag{6.34}$$

with $\widetilde{B}_0 + \lambda N_1 \in \mathfrak{sl}(2,K)$, for

$$\widetilde{X}_1 = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \quad \text{and} \quad \widetilde{B}_0 := \begin{pmatrix} 0 & \frac{1}{w} \\ -q_1 w & 0 \end{pmatrix}.$$

Its corresponding second symmetric power system is

$$[\mathfrak{sym}^2(\widetilde{B}_\lambda)] :\ \widetilde{Y}_1' = -(\widetilde{\widehat{S}}_2 + \lambda \widehat{N}_2)\widetilde{Y}_1, \tag{6.35}$$

where $\widetilde{Y}_1 := \mathrm{Sym}^2(\widetilde{X}_1) = (u_1^2, 2u_1u_2, u_2^2)^t$ and $\widetilde{\widehat{S}}_2 + \lambda\widehat{N}_2 = \mathfrak{sym}^2(\widetilde{B}_0 + \lambda N_1) \in \mathfrak{sl}(3,K)$, for

$$\widetilde{\widehat{S}}_2 := \begin{pmatrix} 0 & -\frac{1}{w} & 0 \\ 2\,wq_1 & 0 & \frac{-2}{w} \\ 0 & wq_1 & 0 \end{pmatrix},$$

where u_1 and q_1 have the explicit form expressed in Theorem 6.3.

Now, consider the second symmetric power system (6.32). Since

$$\widetilde{Y}_1 = \mathrm{Sym}^2(\widetilde{X}_1) = \mathrm{Sym}^2(RP_\lambda R^{-1}) \cdot \mathrm{Sym}^2(X_1) = \mathrm{Sym}^2(RP_\lambda R^{-1}) \cdot Y_1,$$

we get that matrix $P_{2,\lambda} := \mathrm{Sym}^2(RP_\lambda R^{-1})$ is a Darboux transformation which sends system (6.32) into system (6.35).

The rest of the theorem is proved following the same argument as in Proposition 6.5. □

Once we have defined the Darboux transformations for second symmetric power systems, we can state the Darboux transformation for $\mathfrak{so}(3,K)$ systems as follows. As we have found two Darboux transformations for second symmetric power systems, we will have two Darboux transformations for $\mathfrak{so}(3,K)$ systems as well: one by means of Theorem 6.13 and the other by Theorem 6.14.

Applying Proposition 6.7, we can transform the second symmetric power system (6.27) into the $\mathfrak{so}(3,K)$ system

$$[\Omega_\lambda] :\ Z' = -(\Omega_0 + \lambda N_3)Z, \tag{6.36}$$

where $Z = QY$ and $-(\Omega_0 + \lambda N_3) = Q'Q^{-1} - Q(S_2 + \lambda N_2)Q^{-1}$ are given by

$$Z = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}, \quad \Omega_0 = \begin{pmatrix} 0 & -ip & -(q+1) \\ ip & 0 & i(q-1) \\ q+1 & -i(q-1) & 0 \end{pmatrix} \quad \text{and} \quad N_3 = r\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & i \\ 1 & -i & 0 \end{pmatrix}.$$

After performing the Darboux transformation (6.29), system (6.27) is transformed into system (6.28), which, again by Proposition 6.7, can be transformed into the $\mathfrak{so}(3,K)$ system

$$[\widetilde{\Omega}_\lambda]: \ \widetilde{Z}' = -(\widetilde{\Omega}_0 + \lambda N_3)\widetilde{Z}, \tag{6.37}$$

where $\widetilde{Z} = Q\widetilde{Y}$ and $-(\widetilde{\Omega}_0 + \lambda N_3) = Q'Q^{-1} - Q(\widetilde{S}_2 + \lambda N_2)Q^{-1}$ for

$$\widetilde{Z} = \begin{pmatrix} \widetilde{\alpha} \\ \widetilde{\beta} \\ \widetilde{\gamma} \end{pmatrix} \quad \text{and} \quad \widetilde{\Omega}_0 = \begin{pmatrix} 0 & -ip & -(q_1+1) \\ ip & 0 & i(q_1-1) \\ q_1+1 & -i(q_1-1) & 0 \end{pmatrix},$$

for q_1 as in Theorem 6.3. Thus, the Darboux transformation (6.29) also induces a transformation in the corresponding $\mathfrak{so}(3,K)$ systems which sends system (6.36) to system (6.37).

Next result proves that this induced transformation is indeed a Darboux transformations for $\mathfrak{so}(3,K)$ systems.

Corollary 6.15 (First DT for $\mathfrak{so}(3,K)$). *Consider the systems*

$$[\Omega_\lambda]: \ Z' = -(\Omega_0 + \lambda N_3)Z \quad \text{and} \quad [\widetilde{\Omega}_\lambda]: \ \widetilde{Z}' = -(\widetilde{\Omega}_0 + \lambda N_3)\widetilde{Z}$$

given by (6.36) *and* (6.37) *respectively. Let $T_{1,\lambda}$ be the matrix defined by*

$$T_{1,\lambda} = QP_{1,\lambda}Q^{-1} = \frac{1}{2\,(z\tau+\nu)} \begin{pmatrix} -\nu^2+\tau^2+z^2-1 & i\,(\nu^2+\tau^2-z^2-1) & 2(\nu\,\tau+z) \\ i\,(\nu^2-\tau^2+z^2-1) & \nu^2+\tau^2+z^2+1 & 2\,i\,(z-\nu\,\tau) \\ 2(\nu\,z+\tau) & -2\,i\,(\nu\,z-\tau) & 2(\nu-z\tau) \end{pmatrix},$$

where matrix Q is defined by (6.21), *matrix $P_{1,\lambda}$ is defined by expression* (6.29) *and $\nu = \lambda r - z\tau$.*

Then, $T_{1,\lambda}$ is a Darboux transformation, i.e., a virtually strong isogaloisian gauge transformation, which sends system $[\Omega_\lambda]: Z' = -(\Omega_0 + \lambda N_3)Z$ to a system $[\widetilde{\Omega}_\lambda]: \widetilde{Z}' = -(\widetilde{\Omega}_0 + \lambda \widetilde{N}_3)\widetilde{Z}$ of the same shape.

Moreover, when $z \in K$, this Darboux transformation is strong isogaloisian.

Proof. The proof follows applying Proposition 6.7 and Theorem 6.13. In fact, given Y and $\widetilde{Y}$ solutions of equations (6.27) and (6.28) respectively, and Z and $\widetilde{Z}$ solutions of (6.36) and (6.37) respectively, by Proposition 6.7, we have that $Z = QY$ and $\widetilde{Z} = Q\widetilde{Y}$. On the other hand, by Theorem 6.13, we know that $\widetilde{Y} = P_{1,\lambda}Y$. Thus, we get the gauge transformation $\widetilde{Z} = (QP_{1,\lambda}Q^{-1})Z = T_{1,\lambda}Z$. From this, we immediately obtain the gauge transformation for the coefficient matrix:

$$-(\widetilde{\Omega}_0 + \lambda N_3) = T'_{1,\lambda}T^{-1}_{1,\lambda} - T_{1,\lambda}(\Omega_0 + \lambda N_3)T^{-1}_{1,\lambda}.$$

The rest of the corollary is proved following the same argument as in Proposition 6.5. $\square$

Theorem 6.13 and Corollary 6.15 can be summarized in the following commutative diagram:

$$\begin{tikzcd}
{[A_\lambda] : X' = -(A_0 + \lambda N)X} \arrow[r, "P_\lambda"] \arrow[d, "\mathfrak{sym}^2"] & {[\widetilde{A}_\lambda] : \widetilde{X}' = -(\widetilde{A}_0 + \lambda N)\widetilde{X}} \arrow[d, "\mathfrak{sym}^2"] \\
{[\mathfrak{sym}^2(A_\lambda)] : Y' = -(S_2 + \lambda N_2)Y} \arrow[r, "P_{1,\lambda}"] \arrow[d, "Q"] & {[\mathfrak{sym}^2(\widetilde{A}_\lambda)] : \widetilde{Y}' = -(\widetilde{S}_2 + \lambda N_2)\widetilde{Y}} \arrow[d, "Q"] \\
{[\Omega_\lambda] : Z' = -(\Omega_0 + \lambda N_3)Z} \arrow[r, "T_{1,\lambda}"] & {[\widetilde{\Omega}_\lambda] : \widetilde{Z}' = -(\widetilde{\Omega}_0 + \lambda N_3)\widetilde{Z}}
\end{tikzcd}$$

We end this section with the second Darboux transformation for $\mathfrak{so}(3, K)$ systems.

Corollary 6.16 (Second DT for $\mathfrak{so}(3, K)$)**.** *Consider the matrices R defined in* (6.30)*,*

$$R = \begin{pmatrix} 1 & 0 \\ 0 & \frac{1}{w} \end{pmatrix}, \quad \text{and} \quad S = \begin{pmatrix} 1 & 0 & 1 \\ 0 & i & 0 \\ i & 0 & -i \end{pmatrix}.$$

Then, the system $[\widehat{\Omega}_\lambda] : Z_1' = -(\widehat{\Omega}_0 + \lambda \widehat{N}_3)Z_1$, where $Z_1 = S \cdot \mathrm{Sym}^2(RX)$,

$$\widehat{\Omega}_0 = \begin{pmatrix} 0 & i\left(\frac{1}{w} - wq\right) & 0 \\ -i\left(\frac{1}{w} - wq\right) & 0 & \frac{1}{w} + wq \\ 0 & -\left(\frac{1}{w} + wq\right) & 0 \end{pmatrix} \quad \text{and} \quad \widehat{N}_3 = wr \begin{pmatrix} 0 & i & 0 \\ -i & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

is the $\mathfrak{so}(3, K)$ system corresponding to the linear differential equation (6.14)*.*

Moreover, let $T_{2,\lambda}$ be the matrix

$$T_{2,\lambda} = SP_{2,\lambda}S^{-1} = \frac{1}{2r}\begin{pmatrix} z^2 + \frac{\nu^2}{w} + w + \tau^2 & 4i\left(\frac{zw}{1+w} - \frac{\tau\nu}{w}\right) & -i\left(z^2 + \frac{\nu^2}{w} - w - \tau^2\right) \\ i(1+w)\left(\tau - \frac{z\nu}{w}\right) & 2(\lambda r - 2z\tau) & -(1+w)\left(\tau + \frac{z\nu}{w}\right) \\ i\left(z^2 - \frac{\nu^2}{w} + w - \tau^2\right) & -4\left(\frac{zw}{1+w} + \frac{\tau\nu}{w}\right) & z^2 - \frac{\nu^2}{w} - w + \tau^2 \end{pmatrix},$$

where matrix $P_{2,\lambda}$ is defined by expression (6.33) *and $\nu = \lambda r - z\tau$.*

Then, $T_{2,\lambda}$ is a Darboux transformation for system $[\widehat{\Omega}_\lambda]$, i.e., a virtually strong isogaloisian gauge transformation.

Moreover, when $z \in K$, this Darboux transformation is strong isogaloisian.

Proof. Let assume we know $\mathcal{P}_\lambda$ and the associated first order system $X' = A_\lambda X$ defined by (6.14), then we can create an $\mathfrak{so}(3, K)$ system as follows.

First, we set $X_1 := RX$ and $X_1' = B_\lambda X_1$ as in (6.31). Next, consider its second symmetric power system

$$[\mathfrak{sym}^2(B_\lambda)] : \; Y_1' = -(\widehat{S}_2 + \lambda \widehat{N}_2)Y_1$$

as in (6.32). Recall that $Y_1 = \mathrm{Sym}^2(X_1) = \mathrm{Sym}^2(RX)$. Then, we define $Z_1 := S \cdot Y_1$. So, we have $Z_1' = -S(\widehat{S}_2 + \lambda\widehat{N}_2)S^{-1}Z = -(\widehat{\Omega}_0 + \lambda\widehat{N}_3)Z$, where $\widehat{\Omega}_0 = S \cdot \widehat{S}_2 \cdot S^{-1}$ and $\widehat{N}_3 = S \cdot \widehat{N}_2 \cdot S^{-1}$.

Now, since $\widetilde{Y}_1 = P_{2,\lambda} \cdot Y_1$, by Theorem 6.14, it follows that:

$$T_{2,\lambda} = S \cdot P_{2,\lambda} \cdot S^{-1}.$$

Finally, by construction, the image of $\widehat{\Omega}_0 + \lambda\widehat{N}_3$ by this gauge transformation is $\widetilde{\widehat{\Omega}}_0 + \lambda\widehat{N}_3$, where $\widetilde{\widehat{\Omega}}_0$ is obtained from $\widehat{\Omega}_0$ by changing q by the function q_1 obtained in (6.13) by the Darboux transformation, so

$$\widetilde{\widehat{\Omega}}_0 = \begin{pmatrix} 0 & i\left(\frac{1}{w} - wq_1\right) & 0 \\ -i\left(\frac{1}{w} - wq_1\right) & 0 & \frac{1}{w} + wq_1 \\ 0 & -\left(\frac{1}{w} + wq_1\right) & 0 \end{pmatrix}.$$

The rest of the corollary is proved following the same argument as in Proposition 6.5. $\square$

This second construction for Darboux transformations for $\mathfrak{so}(3, K)$ systems can be expressed in the following commutative diagram:

$$\begin{tikzcd}
{[A_\lambda] : X' = -(A_0 + \lambda N)X} \arrow[r, "P_\lambda"] \arrow[d, "R"] & {[\widetilde{A}_\lambda] : \widetilde{X}' = -(\widetilde{A}_0 + \lambda N)\widetilde{X}} \arrow[d, "R"] \\
{[B_\lambda] : X_1' = -(B_0 + \lambda N_1)X_1} \arrow[r, "RP_\lambda R^{-1}"] \arrow[d, "\mathfrak{sym}^2"] & {[\widetilde{B}_\lambda] : \widetilde{X}_1' = -(\widetilde{B}_0 + \lambda N_1)\widetilde{X}_1} \arrow[d, "\mathfrak{sym}^2"] \\
{[\mathfrak{sym}^2(B_\lambda)] : Y_1' = -(\widehat{S}_2 + \lambda\widehat{N}_2)Y_1} \arrow[r, "P_{2,\lambda}"] \arrow[d, "S"] & {[\mathfrak{sym}^2(\widetilde{B}_\lambda)] : \widetilde{Y}_1' = -(\widetilde{\widehat{S}}_2 + \lambda\widehat{N}_2)\widetilde{Y}_1} \arrow[d, "S"] \\
{[\widehat{\Omega}_\lambda] : Z_1' = -(\widehat{\Omega}_0 + \lambda\widehat{N}_3)Z_1} \arrow[r, "T_{2,\lambda}"] & {[\widetilde{\widehat{\Omega}}_\lambda] : \widetilde{Z}_1' = -(\widetilde{\widehat{\Omega}}_0 + \lambda\widehat{N}_3)\widetilde{Z}_1}
\end{tikzcd}$$

Finally, we would like to emphasize here that this approach to the study of orthogonal systems is, as far as we know, different from everything that has been done so far, since until now the problem of obtaining solutions for orthogonal systems was addressed directly. We propose here an indirect study, in which all the results are obtained as a consequence of the computations and constructions made on the second symmetric power differential systems.

Appendix A

Auxiliary results

We establish a series of easy corollaries of the result 3.8. They are necessary in the Subsection 3.1.3. We use the same notations as in Subsection 3.1.2.

Corollary A.1. *We have*

$$F_n^0 F_{n,x} - F_{n,x}^0 F_n = (E - E_0) P_n,$$

where P_n is a polynomial in E of degree at most $n-1$.

Proof. Since $F_n = \sum_{l=0}^n f_{n-l} E^l$ and $F_n^0 = \sum_{l=0}^n f_{n-l} E_0^l$, we have that

$$\begin{aligned} F_n^0 F_{n,x} - F_{n,x}^0 F_n &= \sum_{i,j=0}^n f_{n-i} f_{n-j,x} E_0^i E^j - \sum_{i,j=0}^n f_{n-i} f_{n-j,x} E_0^j E^i \\ &= \sum_{\substack{i,j=0 \\ i \neq j}}^n (E_0^i E^j - E_0^j E^i) f_{n-i} f_{n-j,x}. \end{aligned} \tag{A.1}$$

We factor the term $E_0^i E^j - E_0^j E^i$:

$$E_0^i E^j - E_0^j E^i = (E - E_0)(E E_0)^{\min(i,j)} (-1)^{\operatorname{sign}(i,j)} \left(\sum_{k=0}^{|j-i|-1} E^k E_0^{|j-i|-1-k} \right),$$

and replace it in (A.1). We get

$$\begin{aligned} & F_n^0 F_{n,x} - F_{n,x}^0 F_n = \\ & \quad = (E - E_0) \sum_{\substack{i,j=0 \\ i \neq j}}^n (E E_0)^{\min(i,j)} (-1)^{\operatorname{sign}(i,j)} \left(\sum_{k=0}^{|j-i|-1} E^k E_0^{|j-i|-1-k} \right) f_{n-i} f_{n-j,x} \\ & \quad = (E - E_0) P_n, \end{aligned} \tag{A.2}$$

for P_n a polynomial in E of degree at most $n-1$, as it is stated. ∎

Corollary A.2. *We have*

$$\mu^2(F_n^0)^2 - \mu_0^2 F_n^2 = (E-E_0)\left(\frac{F_nF_n^0P_{n,x}}{2} + F_n^2(F_n^0)^2 - \frac{P_n(F_nF_{n,x}^0 + F_{n,x}F_n^0)}{4}\right),$$

where P_n is the polynomial obtained in Corollary A.1.

Proof. By (3.13) we have

$$\begin{aligned}\mu^2 = R_{2n+1} &= \frac{F_nF_{n,xx}}{2} - (u-E)F_n^2 - \frac{F_{n,x}^2}{4},\\ \mu_0^2 = R_{2n+1}(E_0) &= \frac{F_n^0F_{n,xx}^0}{2} - (u-E_0)(F_n^0)^2 - \frac{(F_{n,x}^0)^2}{4}.\end{aligned}$$

Hence,

$$\begin{aligned}\mu^2(F_n^0)^2 - \mu_0^2F_n^2 &= \frac{F_nF_n^0}{2}(F_{n,xx}F_n^0 - F_{n,xx}^0F_n) + (E-E_0)F_n^2(F_n^0)^2\\ &+ \frac{F_n^2(F_{n,x}^0)^2 - F_{n,x}^2(F_n^0)^2}{4}\\ &= \frac{F_nF_n^0}{2}(F_{n,xx}F_n^0 - F_{n,xx}^0F_n) + (E-E_0)F_n^2(F_n^0)^2\\ &+ \frac{(F_nF_{n,x}^0 - F_{n,x}F_n^0)(F_nF_{n,x}^0 + F_{n,x}F_n^0)}{4}.\end{aligned}$$

As $F_n^0F_{n,xx} - F_{n,xx}^0F_n = (F_n^0F_{n,x} - F_{n,x}^0F_n)_x = (E-E_0)P_{n,x}$, by Corollary A.1 we obtain

$$\mu^2(F_n^0)^2 - \mu_0^2 F_n^2 = (E-E_0)\left(\frac{F_nF_n^0P_{n,x}}{2} + F_n^2(F_n^0)^2 - \frac{P_n(F_nF_{n,x}^0 + F_{n,x}F_n^0)}{4}\right).$$

□

Now, let (E_0,μ_0) be a regular point of Γ_n and $\mu_0 = 0$. In this case, we have that $R_{2n+1}^0 = R_{2n+1}(E_0) = 0$ and $\partial_E(R_{2n+1})(E_0) \neq 0$, thus,

$$\mu^2 = R_{2n+1}(E) = (E-E_0)M_{2n}, \tag{A.3}$$

where $M_{2n}(E)$ is a polynomial in E of degree $2n$ such that $M_{2n}(E_0) \neq 0$.

Corollary A.3. *Let (E_0,μ_0) be a regular point of Γ_n and $\mu_0 = 0$. We have that*

$$\frac{M_{2n}}{F_n} + \frac{(E-E_0)P_n^2}{4F_n(F_n^0)^2}$$

is a polynomial in E of degree n, with P_n the polynomial obtained in Corollary A.1 and M_{2n} the polynomial defined in (A.3).

Proof. We have

$$M_{2n} = \frac{\mu^2}{E-E_0} = \frac{F_n F_{n,xx}}{2(E-E_0)} - \frac{(u-E)F_n^2}{E-E_0} - \frac{F_{n,x}^2}{4(E-E_0)},$$
$$P_n^2 = \frac{(F_n^0 F_{n,x} - F_{n,x}^0 F_n)^2}{(E-E_0)^2} = \frac{(F_n^0)^2 F_{n,x}^2 + (F_{n,x}^0)^2 F_n^2 - 2F_n^0 F_n F_{n,x} F_{n,x}^0}{(E-E_0)^2}.$$

We replace these expressions in the formula and we get:

$$\frac{M_{2n}}{F_n} + \frac{(E-E_0)P_n^2}{4F_n(F_n^0)^2} = \frac{2(F_n^0)^2 F_{n,xx} - 4(u-E)(F_n^0)^2 F_n + (F_{n,x}^0)^2 F_n - 2F_n^0 F_{n,x}^0 F_{n,x}}{4(E-E_0)(F_n^0)^2}.$$

The numerator of this function is a polynomial in E of degree $n+1$ and has a root in $E = E_0$ as can be easily verified replacing E by E_0:

$$2(F_n^0)^2 F_{n,xx}^0 - 4(u-E^0)(F_n^0)^3 - (F_{n,x}^0)^2 F_n^0 = 4F_n^0 \mu_0^2 = 0,$$

by (3.13). So, we get that

$$2(F_n^0)^2 F_{n,xx} - 4(u-E)(F_n^0)^2 F_n + (F_{n,x}^0)^2 F_n - 2F_n^0 F_{n,x}^0 F_{n,x} = (E-E_0)Q_n,$$

where Q_n denotes a polynomial in E of degree n. Hence

$$\frac{M_{2n}}{F_n} + \frac{(E-E_0)P_n^2}{4F_n(F_n^0)^2} = \frac{Q_n}{4(F_n^0)^2}$$

and then the result follows. □

Next, let (E_0, μ_0) be a singular point of Γ_n. In this case, $\mu_0 = 0$, $R_{2n+1}^0 = R_{2n+1}(E_0) = 0$ and $\partial_E(R_{2n+1})(E_0) = 0$, thus,

$$\mu^2 = R_{2n+1}(E) = (E-E_0)^2 Z_{2n-1}, \tag{A.4}$$

where $Z_{2n-1}(E)$ is a polynomial in E of degree $2n-1$ such that $Z_{2n-1}(E_0) \neq 0$.

Corollary A.4. *Let (E_0, μ_0) be a singular point of Γ_n. We have that*

$$\frac{Z_{2n-1}}{F_n} + \frac{P_n^2}{4F_n(F_n^0)^2}$$

is a polynomial in E of degree $n-1$, with P_n the polynomial obtained in Corollary A.1 and Z_{2n-1} the polynomial defined in (A.4)*.*

Proof. It follows by an analogous computation to that of Corollary A.3. □

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